Curvature Inequalities for Operators in the Cowen-Douglas Class of a Planar Domain

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Gadadhar Misra at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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Dedicated to my Parents and Teachers

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Abstract

Fix a bounded planar domain Ω. If an operator T, in the Cowen-Douglas class $B_1(\Omega)$, admits the compact set $\overline{\Omega}$ as a spectral set, then the curvature inequality $\mathcal{K}_T(w) \le -4\pi^2 S_\Omega(w,w)^2$, where S_Ω is the Sžego kernel of the domain Ω, is evident. In particular, for any contraction T in $B_1(\mathbb{D})$, $\mathcal{K}_T(w) \le -4\pi^2 S_{\mathbb{D}}(w,w)^2 = -(1-|w|^2)^{-2}$. The curvature of the unilateral backward shift operator U_+^* equals $(1-|w|^2)^{-2}$, $w \in \mathbb{D}$. However, it is easy to construct examples of contractive operators T in $B_1(\mathbb{D})$ for which $\mathcal{K}_T(w_0) = (1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$ but T is not unitarily equivalent to U_+^*.

After imposing some “mild" conditions on the class of co-subnormal contractions T in $B_1(\mathbb{D})$, it is shown that if $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for an arbitrary but fixed point $w_0 \in \mathbb{D}$, then T is unitarily equivalent to U_+^*.

Except when Ω is simply connected, the existence of an operator for which $\mathcal{K}_T(w) = -4\pi^2 S_\Omega(w,w)^2$ for all w in Ω is not known. However, one knows that if w is a fixed but arbitrary point in Ω, then there exists a bundle shift of rank 1, say S, depending on this w, such that $\mathcal{K}_{S^*}(w) = -4\pi^2 S_\Omega(w,w)^2$. It is proved that these *extremal* operators are uniquely determined: If T_1 and T_2 are two operators in $B_1(\Omega)$ each of which is the adjoint of a rank 1 bundle shift and $\mathcal{K}_{T_1}(w) = -4\pi^2 S_\Omega(w,w)^2 = \mathcal{K}_{T_2}(w)$ for some fixed w in Ω, then T_1 and T_2 are unitarily equivalent. A surprising consequence is that the adjoint of only some of the bundle shifts of rank 1 occur as extremal operators in domains of connectivity ≥ 1. These are then described explicitly.

For a tuple of commuting operator $\boldsymbol{T} = (T_1, \dots, T_m)$ in $B_n(\Omega)$, where Ω is a bounded domain in $\mathbb{C}^m$, a curvature inequality is found.

The module tensor product of a Hilbert module $\mathcal{H}$ a two dimensional Hilbert module over the function algebra $\mathcal{O}(\overline{\Omega})$ given explicitly. In the case of planar domain Ω, using the module tensor product, the dilation for every two dimensional contractive module over $\mathcal{O}(\overline{\Omega})$ is described. The question of explicitly describing the dilations for two dimensional modules over the algebra $\mathcal{O}(\overline{\Omega})$, for any domain $\Omega \subset \mathbb{C}^m$ is also investigated.

Contents

Chapter 1

Introduction

Let Ω be a bounded, open and connected subset of the complex plane $\mathbb{C}$. Assume that $\partial\Omega$, the boundary of Ω, consists of $n+1$ analytic Jordan curves. Set $\Omega^* = \{\bar{z} \mid z \in \Omega\}$, which is again a planar domain whose boundary consists of $n+1$ analytic Jordan curves. Here we study operators in $B_n(\Omega^*)$, first introduced by Cowen and Douglas in the very influential paper [13].

Definition 1.1. An operators T acting on a complex separable Hilbert space $\mathscr{H}$ with its spectrum $\sigma(T)$ contained in Ω^*, is said to be in the class $B_n(\Omega^*)$ if it meets the following requirements.

1. $\operatorname{ran}(T-w) = \mathscr{H}, \ \ w \in \Omega^*$,
2. $\bigvee_{w\in\Omega^*} \ker(T-w) = \mathscr{H}$ and
3. $\dim(\ker(T-w)) = n, \ \ w \in \Omega^*$.

These conditions ensure the existence of a rank n Hermitian holomorphic vector bundle E_T over Ω^*, that is,

$$E_T := \{(w, v) \in \Omega^* \times \mathscr{H} : v \in \operatorname{Ker}(T-w)\}, \pi(w, v) = w,$$

and there exist a holomorphic frame $w \mapsto \gamma(w) := (\gamma_1(w), \gamma_2(w), \dots, \gamma_n(w))$ with the property ker(T − w) = span $\{\gamma_i(w) : 1 \le i \le n\}$ (cf. [13, Proposition 1.11]). Consequently the function $\Theta : \Omega^* \mapsto \operatorname{Gr}(\mathscr{H}, n)$, $\Theta(w) = \ker(T-w)$ is holomorphic, where $\operatorname{Gr}(\mathscr{H}, n)$ is the Grassmannian manifold of n - dimensional subspaces in the Hilbert space $\mathscr{H}$. So the holomorphic Hermitian bundle E_T is the pull-back of the tautological bundle on $\operatorname{Gr}(\mathscr{H}, n)$ under Θ. Cowen and Douglas proved that the equivalence class of Hermitian holomorphic bundle E_T and the unitary equivalence class of the operator T determine each other (cf. [13, Theorem 1.14]).

Theorem 1.2 (Cowen-Douglas)**.** *Two operators T_1 and T_2 in $B_n(\Omega^*)$ are unitarily equivalent if and only if the associated Hermitian holomorphic vector bundles E_{T_1} and E_{T_2} are locally equivalent.*

The Hermitian structure of the vector bundle E_T at the point w with respect to the frame $\gamma(w)$ is obtained from that of the subspace $\ker(T-w)$ of the Hilbert space $\mathscr{H}$ and we denote it by $h(w) = \big(\!\big(\langle \gamma_j(w), \gamma_i(w)\rangle_{\mathscr{H}}\big)\!\big)$. The curvature K_γ of the bundle E_T w.r.t the frame γ is given by the following formula (see [43, Proposition 2.2])

$$\begin{aligned}\mathsf{K}_\gamma(w) &= \frac{\partial}{\partial \bar{w}}\big(h^{-1}(w)\frac{\partial}{\partial w}h(w)\big)d\bar{w}\wedge dw\\ &= \mathscr{K}_\gamma(w)dw\wedge d\bar{w}.\end{aligned}$$

We will not distinguish between the $(1,1)$ - form $\mathscr{K}_\gamma(w)d\bar{w}\wedge dw$ and the co-efficient $(n\times n)$ matrix $\mathscr{K}_\gamma(w)$. Both of these, depending on the context, will be called the curvature of the vector bundle E_T. The curvature of a vector bundle with respect to two frames transforms according the rule

$$\mathscr{K}_{\gamma g}(w) = g^{-1}(w)\mathscr{K}_\gamma(w)g(w),\ w\in\Omega_0,$$

where $\gamma = (\gamma_1,\dots,\gamma_n)$ is a frame for E_T over an open subset $\Omega_0 \subseteq \Omega^*$ and $g:\Omega_0 \mapsto GL_n(\mathbb{C})$ is a change of frame. If the rank of the bundle is greater than 1, although the curvature $\mathscr{K}_\gamma(w)$ depends on the frame γ, eigenvalues of the curvature are invariant for the bundle E_T. Cowen and Douglas have shown that a complete set of invariants for the bundle E_T involves curvature and a number of its covariant derivatives,

$$\mathsf{K}_{z^i\bar{z}^j},\ 0\le i\le j\le i+j\le n,\ \big((i,j)\ne(n,0),(0,n)\big),$$

where rank of E_T equal to n (cf. [13, Theorem 3.17]). The case of rank 1 is very special. In this case, the curvature of the bundle E_T w.r.t the frame γ is of the form

$$\mathscr{K}_\gamma(w) = -\frac{\partial^2}{\partial w\partial\bar{w}}\log\|\gamma(w)\|^2$$

and the definition of the curvature is independent of the choice of the frame γ. In fact if $\tilde{\gamma}$ is another frame for E_T, then $\tilde{\gamma} = \phi\gamma$ for some non vanishing holomorphic function ϕ and the harmonicity of the $\log|\phi(w)|$ gives $\mathscr{K}_\gamma(w) = \mathscr{K}_{\tilde{\gamma}}(w)$. So, in the case of a line bundle bundle E_T, we simply denote the curvature by $\mathscr{K}(w)$ as it's definition is independent of the choice of the frame. We also denote the curvature by $\mathscr{K}_T(w)$ and call it curvature for the operator T.

Let U_+^* be the backward shift operator on the Hilbert space $\ell^2(\mathbb{N})$. For w in the unit disc $\mathbb{D}$, the vector $\gamma(w) := (1,w,w^2,\dots,w^p,\dots)$ is in $\ell^2(\mathbb{N})$ and $U_+^*(\gamma(w)) = w\gamma(w)$. In this case, it is not hard to see that $\dim\ker(U_+^*-w) = 1$ and (U_+^*-w) is an onto linear map for all $w\in\mathbb{D}$.

Consequently, γ defines a holomorphic curve in $\mathrm{Gr}(\ell^2(\mathbb{N}),1)$. The corresponding holomorphic Hermitian vector bundle on $\mathbb{D}$ is the trivial bundle with the metric $\|\gamma(w)\|^2 = (1-|w|^2)^{-1}$ at w. Now an easy computation shows that $\mathscr{K}_{U_+^*}(w) = -(1-|w|^2)^{-2},\ w \in \mathbb{D}$.

There is a very direct relationship between the operator T in $B_1(\Omega^*)$ and the curvature $\mathscr{K}_T(w)$. This is best described in terms of the local operators $T_{|\ker(T-w)^2}$. Indeed, the two vectors $\gamma(w), \gamma'(w)$ span $\ker(T-w)^2$. With respect to the orthonormal basis obtained from these two vectors using the Gram-Schmidt process, we have the matrix representation for the local operator

$$T_{|\ker(T-w)^2} = \begin{pmatrix} w & (-\mathscr{K}_T(w))^{-1/2} \\ 0 & w \end{pmatrix},\ w \in \Omega^*.$$

Cowen and Douglas have shown that the local operators $T_{|\ker(T-w)^2}$, $w \in \Omega^*$, altogether determine the unitary equivalence class for an operator $T \in B_1(\Omega^*)$ (cf. [13, Theorem 1.6]). Consequently the curvature $\mathscr{K}_T(w)$ is a complete invariant for the rank 1 bundle E_T or equivalently for the unitary equivalence class of operator T.

Theorem 1.3 (Cowen-Douglas)**.** *Two operators T and $\tilde{T}$ in $B_1(\Omega^*)$ are unitarily equivalent if and only if the local operators $T_{|\ker(T-w)^2}$ and $\tilde{T}_{|\ker(\tilde{T}-w)^2}$ are unitarily equivalent for all $w \in \Omega^*$. Equivalently, T and $\tilde{T}$ are unitarily equivalent if and only if $\mathscr{K}_T(w) = \mathscr{K}_{\tilde{T}}(w)$ for all $w \in \Omega^*$.*

Finally, Cowen and Douglas also provided a model for the operators in the class $B_1(\Omega^*)$, which is easy to describe:

If $T \in B_1(\Omega^*)$ then T is unitarily equivalent to the adjoint M^* of the operator of multiplication M by the coordinate function on some Hilbert space $\mathscr{H}_K$ consisting of holomorphic function on Ω possessing a reproducing kernel K. From now on, we let M denote the operator of multiplication by the coordinate function and as usual M^* denotes its adjoint.

The kernel K is complex valued function defined on $\Omega \times \Omega$, which is holomorphic in the first and anti-holomorphic in the second variable and is positive definite in the sense that $\big(\!\big(K(z_i,z_j)\big)\!\big)$ is positive definite for every subset $\{z_1,\ldots,z_n\}$ of the domain Ω. We will therefore assume, without loss of generality, that an operator T in $B_1(\Omega^*)$ has been realized as the operator M^* on some reproducing kernel Hilbert space $\mathscr{H}_K$. Since $\bar{w} \mapsto K(\cdot,w)$ is a frame for the bundle E_T over the domain Ω^*, the curvature $\mathscr{K}_T(\bar{z})$ can be computed using kernel function.

$$\mathscr{K}_T(\bar{z}) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w)|_{w=z} = -\frac{\|K_z\|^2 \|\bar{\partial} K_z\|^2 - |\langle K_z, \bar{\partial} K_z\rangle|^2}{(K(z,z))^2},\ z \in \Omega,$$

where K_z and $\bar{\partial} K_z$ are the vectors

$$K_z(u) := K(u,z)\ ,u \in \Omega,$$

$$\bar{\partial} K_z(u) := \frac{\partial}{\partial \bar{w}} K(u,w)\,|_{w=z}\ ,u \in \Omega,$$

in $\mathscr{H}_K$. Thus the curvature $\mathscr{K}_T(w)$ is a real analytic function on Ω^*. As we have seen earlier that $\mathscr{K}_T(w)$ is independent of the choice of a frame of E_T, it follows that expression for $\mathscr{K}_T(w)$ in terms of kernel function K is independent of the representation of T as the operator M^* on some reproducing kernel Hilbert space $\mathscr{H}_K$ possessing K as a reproducing kernel. Indeed, if T also admits a representation as the adjoint of the multiplication operator on another reproducing kernel Hilbert space $\mathscr{H}_{\tilde{K}}$, then we must have $K(z,w) = \varphi(z)\tilde{K}(z,w)\overline{\varphi(w)}$ for some non vanishing holomorphic function φ defined on Ω, see [13, Section 1.15] . This implies $\frac{\partial^2}{\partial w \partial \bar{w}} \log K(w,w)|_{w=z} = \frac{\partial^2}{\partial w \partial \bar{w}} \log \tilde{K}(w,w)|_{w=z}$.

Normalized kernel: For any fixed but arbitrary $\zeta \in \Omega$, the function $K(z,\zeta)$ is non-zero in some neighbourhood, say U, of ζ. The function $\varphi_\zeta(z) := K(z,\zeta)^{-1}K(\zeta,\zeta)^{1/2}$ is then holomorphic. The linear space $(\mathscr{H}, K_{(\zeta)}) := \{\varphi_\zeta f : f \in \mathscr{H}_K\}$ then can be equipped with an inner product making the multiplication operator M_{φ_ζ} unitary. It then follows that $(\mathscr{H}, K_{(\zeta)})$ is a space of holomorphic functions defined on $U \subseteq \Omega$, it has a reproducing kernel $K_{(\zeta)}$ defined by

$$K_{(\zeta)}(z,w) = K(\zeta,\zeta)K(z,\zeta)^{-1}K(z,w)\overline{K(w,\zeta)}^{-1},\ z,w \in U,$$

with the property $K_{(\zeta)}(z,\zeta) = 1, z \in U$ and finally the multiplication operator M on $\mathscr{H}_K$ is unitarily equivalent to the multiplication operator M on $(\mathscr{H}, K_{(\zeta)})$. The kernel $K_{(\zeta)}$ is said to be normalized at ζ.

The realization of an operator T in $B_1(\Omega^*)$ as the adjoint of the multiplication operator on $\mathscr{H}_K$ is not canonical. The kernel function K is determined only upto conjugation by a holomorphic function. Consequently, one sees that the curvature $\mathscr{K}_k$ is unambiguously defined. on the other hand, Curto and Salinas (cf. [16, Remarks 4.7 (b)]) prove that the multiplication operators M on the two Hilbert spaces $(\mathscr{H}, K_{(\zeta)})$ and $(\hat{\mathscr{H}}, \hat{K}_{(\zeta)})$ are unitarily equivalent if and only if $K_{(\zeta)} = \hat{K}_{(\zeta)}$ in some small neighbourhood of ζ. Thus the normalized kernel at ζ, that is, $K_{(\zeta)}$ is also unambiguously defined. It follows that the curvature and the normalized kernel at ζ serve equally well as a complete unitary invariant for the operator T in $B_1(\Omega^*)$.

Since the unitary equivalence class of the local operators $T_{|\ker(T-w)^2}$, or equivalently, the curvature $\mathscr{K}_T(w)$ is a complete invariant for the unitary equivalence class of T, it is natural to study how other properties of the local operators, or equivalently, the curvature $\mathscr{K}_T(w)$ are related to those of the operator T. For instance, if T is a contraction in $B_1(\mathbb{D})$, then the local operator $T_{|\ker(T-w)^2}$ is also a contraction for each $w \in \mathbb{D}$. Now, using the matrix representation of the local operator, we conclude that $\mathscr{K}_T(w) \le -(1-|w|^2)^{-2}$. This inequality for the curvature can be generalised considerably. Recall that a compact subset $X \subseteq \mathbb{C}$ is said to be a spectral set for an operator A in $\mathscr{L}(\mathscr{H})$, if

$$\sigma(A) \subseteq X \ \text{and}\ \sup\{\|r(A)\| : r \in \mathrm{Rat}(X) \ \text{and}\ \|r\|_\infty \le 1\} \le 1,$$

where Rat(X) denotes the algebra of rational function whose poles are off X and $\|r\|_\infty$ denotes the sup norm over the compact subset X. Equivalently, X is a spectral set for the operator A if the homomorphism $\rho_A : \mathrm{Rat}(X) \to \mathscr{L}(\mathscr{H})$ defined by the formula $\rho_A(r) = r(A)$ is contractive. There are plenty of examples where the spectrum of an operator is a spectral set. This is the case for subnormal operators, see [23, Chapter 21].

Curvature Inequality: For every operator T in $B_1(\Omega^*)$, which admits $\overline{\Omega}^*$ as a spectral set, we have

$$\mathscr{K}_T(\bar{w}) \leq -4\pi^2 (S_{\Omega^*}(\bar{w}, \bar{w}))^2, \quad \bar{w} \in \Omega^*, \tag{1.1}$$

where S_{Ω^*} is the Sz̈ego kernel of the domain Ω^*.

In the following chapters, we will study (i) briefly the case when Ω is the unit disc $\mathbb{D}$, (ii) mainly the case when Ω is a finitely connected bounded planar domain and (iii) finally, the case when Ω is a bounded domain in $\mathbb{C}^m$. We recall the preliminaries, closely following [2], [20], in the case of a planar domain that we will be using throughout.

Definition 1.4 (Hardy space)**.** Let Ω be a bounded domain in $\mathbb{C}$ and $\mathscr{O}(\Omega)$ be the space of holomorphic functions on Ω. The Hardy space on Ω is defined to be the linear space

$$H^2(\Omega) := \{f \in \mathscr{O}(\Omega) : |f(z)|^2 \leq v_f(z), \text{ for some harmonic function } v_f(z) \text{ on } \Omega\}.$$

As every function f in $H^2(\Omega)$ admits a harmonic majorant $v_f(z)$, it also admits a least harmonic majorant, say $u_f(z)$, on Ω. Fix a point $p \in \Omega$. It is easily verified that $\|f\|^2 := u_f(p)$ defines a norm on $H^2(\Omega)$ and makes it into a Hilbert space. It is known that different choice of $p \in \Omega$ induces an equivalent norm on $H^2(\Omega)$ (cf. [20, Ch.3, Theorem 2.1]).

Let $d\omega_p$ be the harmonic measure relative to the point $p \in \Omega$ and let $(L^2(\partial\Omega), d\omega_p)$ denote the space of square integrable functions defined on $\partial\Omega$ with respect to the measure $d\omega_p$. The closed subspace

$$(H^2(\partial\Omega), d\omega_p) := \left\{f \in (L^2(\partial\Omega), d\omega_p) : \int_{\partial\Omega} f g \, d\omega_p = 0, g \in \mathscr{O}(\overline{\Omega})\right\},$$

where $\mathscr{O}(\overline{\Omega})$ is the space of all functions which are holomorphic in some open neighborhood of the closed set $\overline{\Omega}$, is the Hardy space of $\partial\Omega$.

A function f in $H^2(\Omega)$ admits a boundary value $\hat{f}$. This means that the $\lim_{z\to\lambda} f(z)$ exists (almost everywhere relative to $d\omega_p$) as z approaches $\lambda \in \partial\Omega$ through any non-tangential path in Ω. Define the function $\hat{f} : \partial\Omega \to \mathbb{C}$ by setting $\hat{f}(\lambda) = \lim_{z\to\lambda} f(z)$. It then follows that the map $f \mapsto \hat{f}$ is an isometric isomorphism between the two hardy spaces $H^2(\Omega)$ and $(H^2(\partial\Omega), d\omega_p)$ (cf. [20, Ch 4, Theorem 4.4]). Because of this correspondence, we often denote the Hardy space by $(H^2(\Omega), d\omega_p)$.

Sz̈ego kernel: The harmonic measure $d\omega_p$ on $\partial\Omega$ is boundedly mutually absolutely continuous w.r.t the arc length measure ds on $\partial\Omega$. The norm given by $\|g\|^2 = \int_{\partial\Omega} |g|^2 \, ds$, for $g \in L^2(\partial\Omega, d\omega_p)$ will then define an equivalent norm on $L^2(\partial\Omega)$. We denote the the subspace

$H^2(\partial\Omega)$, endowed with this new norm, by $(H^2(\partial\Omega), ds)$. Consequently, using the isomorphism between $H^2(\Omega)$ and $H^2(\partial\Omega)$, we have an equivalent norm on $H^2(\Omega)$ making it into a Hilbert space, which we denote by $(H^2(\Omega), ds)$, so that the map $f \mapsto \hat{f}$ is an isometric isomorphism between the two Hilbert spaces $(H^2(\Omega), ds)$ and $(H^2(\partial\Omega), ds)$. Since $(H^2(\Omega), d\omega_p)$ is a reproducing kernel Hilbert space(cf. [20, Ch 3, Proposition 2.4]), it follows that the Hilbert space $(H^2(\Omega), ds)$ is also a reproducing kernel Hilbert space. The reproducing kernel associated to $(H^2(\Omega), ds)$ is called the Sz̈ego kernel and is denoted by $S_\Omega(z, w)$, for $z, w \in \Omega$.

Flat unitary bundle and bundle shift operator: Let α be an element in $\mathrm{Hom}(\pi_1(\Omega), \mathbb{T})$, that is, it is a homomorphism from the fundamental group $\pi_1(\Omega)$ of Ω into the unit circle $\mathbb{T}$. Such homomorphism is also called a character. Each of these character induce a flat unitary bundle E_α of rank 1 on Ω and conversely every flat unitary bundle of rank 1 on the domain Ω is equivalent to one such bundle E_α for some character α (cf. [9, Pg. 850-851]). Following theorem establishes one to one correspondence between $\mathrm{Hom}(\pi_1(\Omega), \mathbb{T})$ and the set of equivalence classes of flat unitary vector bundle over Ω of rank 1 (cf. [22, p. 186].)

Theorem 1.5. *Two rank one flat unitary vector bundle E_α and E_β are equivalent as flat unitary vector bundle if and only if their inducing characters are equal that is $\alpha = \beta$.*

Let E_α be a flat unitary vector bundle of rank one over the domain Ω. If f is a holomorphic section of the bundle E_α, then for $z \in U_i \cap U_j$, where $\{U_i, \phi_i\}_{i\in I}$ is a coverng of Ω, we have that $|(\phi_i^z)^{-1}(f(z))| = |(\phi_j^z)^{-1}(f(z))|$. Thus the function $h_f(z) := |(\phi_i^z)^{-1}(f(z))|$, $z \in U_i$, is well defined on all of Ω and is subharmonic there. Let $H^2_{E_\alpha}$ be the linear space of those holomorphic sections f of E_α such that the subharmonic function $(h_f)^2$ on Ω is majorized by a harmonic function on Ω. Let u_f^E be the least harmonic majorant for a section $f \in H^2_{E_\alpha}$. As before, fixing a point p in Ω, the norm defined by $\|f\|^2_{E_\alpha,p} := u_f^E(p)$ on $H^2_E(\Omega)$ makes it into a Hilbert space. A bundle shift T_{E_α} is simply the operator of multiplication by the coordinate function on $H^2_{E_\alpha}$.

Like the usual Hardy space, in this case also, every section $f \in H^2_{E_\alpha}$ admits boundary value $\hat{f}$ (almost every where relative to $d\omega_p$)and the map $f \mapsto \hat{f}$ is a linear isomorphism between $H^2_{E_\alpha}$ and its boundary value. In fact it can be shown that

$$\|f\|^2 = \int_{\partial\Omega} |\hat{f}(z)|^2\, d\omega_p(z).$$

We often denote the Hilbert space $H^2_{E_\alpha}$ by $(H^2_{E_\alpha}, d\omega_p)$ to clarify the norm. It is also known that different choices of p in Ω will induce an equivalent norm $H^2_{E_\alpha}$.

Theorem 1.6 (Abrahamse and Douglas)**.** *Let E_α and E_β be two rank one flat unitary vector bundles induced by the homomorphisms α and β respectively. Then the bundle shift T_{E_α} on $(H^2_{E_\alpha}, d\omega_p)$ is unitarily equivalent to the bundle shift T_{E_β} on $(H^2_{E_\beta}, d\omega_p)$ if and only if E_α and E_β are equivalent as flat unitary vector bundles.*

An operator T in $\mathscr{L}(\mathscr{H})$ is said to be rationally cyclic if there exist a vector v_0 in $\mathscr{H}$ such that $\{r(T)(v_0) \mid r \in \mathrm{Rat}(\overline{\Omega})\}$ is dense in $\mathscr{H}$. It is not very hard to verify that T_{E_α} is a pure, rationally cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum $\partial\Omega$. In fact these are the characterizing property for the rank one bundle shift.

Theorem 1.7 (Abrahamse and Douglas)**.** *Every pure, rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and the normal spectrum contained in $\partial\Omega$ is unitarily equivalent to a bundle shift T_{E_α} on $(H^2_{E_\alpha}, d\omega_p)$ for some character α.*

If we consider the norm $\|f\|^2 = \int_{\partial\Omega} |\hat{f}(z)|^2 ds$, then it is easy to see that this defines an equivalent norm on $H^2_{E_\alpha}$. We denote the Hilbert space $H^2_{E_\alpha}$ endowed with this norm by $(H^2_{E_\alpha}, ds)$. It follows that the operator T_{E_α} on $(H^2_{E_\alpha}, ds)$ is a rationally cyclic pure subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum equal to $\partial\Omega$. Hence the operator T_{E_α} on $(H^2_{E_\alpha}, ds)$ must be unitarily equivalent to T_{E_β} on $(H^2_{E_\beta}, d\omega_p)$ for some character β. We have two kind of bundle shifts namely $\{T_{E_\alpha}$ on $(H^2_{E_\alpha}, ds) : \alpha \in \mathbb{T}^n\}$ and $\{T_{E_\alpha}$ on $(H^2_{E_\alpha}, d\omega_p) : \alpha \in \mathbb{T}^n\}$. We have established a bijective correspondence between these two family which preserve the unitary equivalence class.

Let $\mathrm{Gr}(H^2_{E_\alpha}(\Omega), 1)$ be the Grassmannian manifold of 1-dimensional subspaces in the Hardy space $H^2_{E_\alpha}(\Omega)$. It can be shown that the map $\kappa : \Omega \to \mathrm{Gr}(H^2_{E_\alpha}(\Omega), 1)$, $\kappa(z) = \ker(T_E - z)^*$ is anti-holomorphic. Pulling back the tautological bundle under the map κ, we get an anti-holomorphic Hermitian vector bundle E_κ on Ω.

As a consequence of the theorem of Abrahamse-Douglas combined with that of Cowen-Douglas, we see that E_κ and F_κ are (locally) equivalent as holomorphic Hermitian vector bundles on Ω if and only if E and F are equivalent as flat unitary vector bundles on Ω.

A natural question is to determine which of the holomorphic Hermitian vector bundles on Ω^* correspond to a flat unitary vector bundle on Ω. In other words, given the holomorphic Hermitian vector bundle E_T, T in $B_1(\Omega^*)$, how to decide if T^* is a bundle shift.

Let $\mathscr{B}[w] := \{f \in \mathscr{O}(\Omega^*) : \|f\|_\infty \leq 1, f(w) = 0\}$. It is well-known that the extremal problem

$$\sup\{|f'(w)| : f \in \mathscr{B}[w]\} \tag{1.2}$$

admits a solution, say, $F_w \in \mathscr{B}[w]$. The function F_w is called the Ahlfor's function and maps Ω^* onto $\mathbb{D}$ in a n to 1 fashion if the connectivity of the region Ω^* is n. Indeed, F_w is a branched covering map and $F'_w(w) = 2\pi S_{\Omega^*}(w,w)$, where $S_{\Omega^*}(w,w)$ is the Szëgo kernel, the reproducing kernel for the Hardy space $(H^2(\Omega^*), ds)$.

The Poincare metric for the trivial holomorphic line bundle on Ω^* is $F'_w(w)\,dw \otimes d\bar{w}$ at $w \in \Omega^*$. We have the curvature inequality

$$-\frac{\partial^2}{\partial w \partial \bar{w}} \log F'_w(w) \leq -F'_w(w)^2, \; w \in \Omega^*.$$

We have equality throughout the domain Ω^* if it is simply connected. However if Ω^* is not simply connected, it has been shown that the inequality is strict for each $w \in \Omega$ (cf. [36]).

Pick any operator T in the Cowen-Douglas class $B_1(\Omega^*)$, which admits $\overline{\Omega}^*$ as a spectral set. Then $\overline{\Omega}^*$ is also a spectral set for the local operators $T_{|\ker(T-w)}$ $w \in \Omega^*$. Representing these operators in the form $\left(\begin{smallmatrix} w & (-\mathscr{K}_T(w))^{-1/2} \\ 0 & w \end{smallmatrix}\right)$ and applying the usual functional calculus, we obtain

$$r\big(T_{|\ker(T-w)^2}\big) = \begin{pmatrix} r(w) & \frac{r'(w)}{\sqrt{-\mathscr{K}_T(w)}} \\ 0 & r(\bar{w}) \end{pmatrix},\ r \in \mathrm{Rat}(\overline{\Omega}^*).$$

Clearly, finding

$$\sup\{\|r\big(T_{|\ker(T-w)^2}\big)\| : r \in \mathrm{Rat}(\overline{\Omega}^*)\ \|r\|_\infty \le 1\}$$

is equivalent to finding a solution to the extremal problem (1.2). Thus for operators T in $B_1(\Omega^*)$ which admit $\overline{\Omega}^*$ as a spectral set, we have the curvature inequality

$$\mathscr{K}_\varrho(w) := -\frac{\partial^2}{\partial w \partial \bar{w}} \log \varrho(w) \le -F'_w(w)^2,\ w \in \Omega^*,$$

where $\varrho(w) = \|\gamma(w)\|^2$ for some non-vanishing holomorphic section γ of the line bundle E_T on Ω^*.

For example, consider the operator $T^*_{E_\alpha}$ on $(H^2_{E_\alpha}, ds)$. It follows from a result of Abrahamse and Douglas that the operator $T^*_{E_\alpha}$ on $(H^2_{E_\alpha}, ds)$ belongs to $B_1(\Omega^*)$ for every character α. Since T_{E_α} is a subnormal operator, Ω^* is a spectral set for the operator $T^*_{E_\alpha}$. Consequently, the curvature of $T^*_{E_\alpha}$ satisfy the above inequality.

Fix $w_0 \in \Omega^*$ and ask if there exists an (extremal) operator T in $B_1(\Omega^*)$ admitting $\overline{\Omega}^*$ for which $\mathscr{K}_\varrho(w_0) = F_{w_0}(w_0)^2$. The answer is that such extremal operators exist. Indeed, for a fixed w_0, there is a rank 1 bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$, whose adjoint is extremal at w_0. However, it is not clear if an extremal operator must be the adjoint of a bundle shift except in the case of a simply connected domain. Indeed, R. G. Douglas had raised the following question in the case of the unit disc:

Question 1.8 (R. G. Douglas). For a contraction T in $B_1(\mathbb{D})$, if $\mathscr{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some fixed w_0 in $\mathbb{D}$, then does it follow that T must be unitarily equivalent to the operator U_+^*?

It is easy to see that the answer is negative by means of examples (cf. [25]). However, in this thesis, it is shown that the question has an affirmative answer within a smaller class of contractions in $B_1(\mathbb{D})$. If Ω is a finitely connected domain, then the question of Douglas takes the form: For an operator T in $B_1(\Omega^*)$ admitting $\overline{\Omega}^*$ as a spectral set, if $\mathscr{K}_T(\bar{w}_0) = S_{\Omega^*}(\bar{w}_0, \bar{w}_0)$, $w_0 \in \Omega$, then does it follow that T must be unitarily equivalent to an extremal operator at w_0? This is a question of uniqueness of the extremal operators. The situation here is more complicated. In this case, it is not obvious that each of the bundle shifts must be extremal for some w in Ω. Therefore it is natural to ask which of the bundle shifts occur as an extremal operator. The unexpected answer that not all of them occur as extremal operators follows from first showing that if w_0 is in Ω^*, then there is a unique bundle shift whose adjoint is an extremal operator at w_0. The proof actually identifies the extremal bundle shifts explicitly.

We now describe our results on dilation of the local operators in the multi-variate context. Let Ω be a bounded domain in $\mathbb{C}^m$. Set $\Omega^* = \{\bar{z} : z \in \Omega\}$. A commuting m- tuple of operators $\boldsymbol{T} = (T_1, \ldots, T_m)$ is said to be in the Cowen-Douglas class $B_1(\Omega^*)$ if the dimension of the joint kernel $\cap_{i=1}^m \ker(T_i - w_i)$ is 1 and there is a vector $\gamma(w) \in \cap_{i=1}^m \ker(T_i - w_i)$ such that the map $w \mapsto \gamma(w)$ is holomorphic. A formal definition is given in Chapter 4. The local operator at w is defined to be the m- tuple:

$$N_i(w) = T_{i|\mathscr{M}_w},\ \mathscr{M}_w = \cap_{\ell,k=1}^m \ker(T_\ell - w_\ell)(T_k - w_k).$$

In the paper [14], it is proved that two such operators in $B_1(\Omega^*)$ are unitarily equivalent if and only if the local operators are unitarily equivalent. As before, the unitary equivalence is determined by the curvature

$$\mathscr{K}_{\boldsymbol{T}}(w) = \partial_i \bar{\partial}_j \log \|\gamma(w)\|^2.$$

The space $\mathscr{M}_w$ is the span of the vectors $\gamma(w), (\partial_1\gamma)(w) \ldots, (\partial_m\gamma)(w)$. Let $v \in \mathbb{C}^m$. Consider the two dimensional subspace $\mathscr{S}_w(v)$ of $\mathscr{M}_w$ spanned by the vectors $\gamma(w)$ and $\partial_v\gamma(w)$. The matrix representation of the restriction of the local operator $N_i(w)$ to $\mathscr{S}_w(v)$, with respect to an orthonormal basis obtained using the Gram-Schmidt process is of the form

$$\begin{pmatrix} w_i & \frac{v_i}{\sqrt{\langle -\mathscr{K}_{\boldsymbol{T}}(w)v, v\rangle}} \\ 0 & w_i \end{pmatrix},\ 1 \leq i \leq m.$$

Clearly, if the commuting m- tuple $\boldsymbol{T}$ admits a $\mathscr{O}(\overline{\Omega}^*)$ boundary dilation, then it serves as a dilation for the local operators. What if we assume that all the local operators admit a boundary dilation, then does it follow that $\boldsymbol{T}$ admits such a dilation as well? The answer, in general, is no. A related question that we study, using the notion of a module tensor product, is to investigate which local operators can be realized as restrictions of some commuting m- tuple in the Cowen-Douglas class.

Given any two Hilbert modules $\mathscr{M}$ and $\mathscr{N}$ over a function algebra $\mathscr{A}(X), X \subset \mathbb{C}^m$, let $\mathscr{S}$ be the submodule generated by the vectors

$$\big\{r \cdot f \otimes g - f \otimes r \cdot g : f \in \mathscr{M}, g \in \mathscr{N}, r \in \mathscr{A}\big\}.$$

The module tensor product $\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}$ is the quotient (Hilbert) module

$$\big(\mathscr{M} \otimes \mathscr{N}\big) \ominus \mathscr{S},$$

again over the algebra $\mathscr{A}$, where the map

$$(r,[f\otimes g]) \to P_{\mathscr{S}^\perp}(r\cdot f\otimes g)_{|\mathscr{S}^\perp},\ f\in\mathscr{M}, g\in\mathscr{N}$$

defines the module map for the quotient module. In other words, for $r\in\mathscr{A}$, the map

$$r\mapsto r(\boldsymbol{M})\otimes I_{\mathscr{N}},\ \text{where}\ (r(\boldsymbol{M})f)=r\cdot f,\ f\in\mathscr{M},$$

is a power dilation of the quotient map. This need not be a normal boundary dilation unless the commuting tuple $\boldsymbol{M}$ is jointly subnormal whose normal spectrum lies in the boundary of Ω. Fixing $\mathscr{M}$ to be a module in the Cowen-Douglas class and taking $\mathscr{N}$ to be a finite dimensional module with different Hermitian structures, we describe what are the quotient modules. This provides an explicit construction of a dilation for a class of finite dimensional modules. We identify those contractive modules which can be realized as the quotient of some module in the Cowen-Douglas class.

Main results of the thesis:

Chapter 2 – Curvature Inequality and the case of unit disc: In this chapter, first we establish the curvature inequality (1.1). In the case of the unit disc $\mathbb{D}$, we know that $-4\pi^2(S_{\mathbb{D}}(\bar w,\bar w))^2 = -(1-|w|^2)^{-2}$. Thus if $\mathscr{K}_T(\bar w) = -4\pi^2(S_{\mathbb{D}}(\bar w,\bar w))^2$ for all w in some open subset of $\mathbb{D}$, then it must be unitarily equivalent to the adjoint of the unilateral backward shift U_+^* by the Cowen-Douglas theorem since $\mathscr{K}_{U_+^*}(\bar w) = -(1-|w|^2)^{-2}$. However if $\mathscr{K}_T(\bar w) = -4\pi^2(S_{\Omega^*}(\bar w,\bar w))^2$ for some fixed w in $\mathbb{D}$, then the question of R. G. Douglas asks if T must be unitarily equivalent to U_+^*. It is easy to see that the answer is negative by means of examples. The theorem below provides an affirmative answer to the question of R. G. Douglas after making some restrictive assumptions. To understand the case of equality at some fixed point in the curvature inequality, the following lemma is useful.

Lemma. *Let T be a contraction in $B_1(\mathbb{D})$ and $\mathscr{H}_K$ be the associated reproducing kernel Hilbert space. Then for an arbitrary but fixed $\zeta\in\mathbb{D}$, we have $\mathscr{K}_T(\bar\zeta) = -\frac{1}{(1-|\zeta|^2)^2}$ if and only if the vectors $\tilde K_\zeta, \bar\partial\tilde K_\zeta$ are linearly dependent in the Hilbert space $\mathscr{H}_{\tilde K}$, where $\tilde K(z,w) = (1-z\bar w)K(z,w)$.*

Applying the preceding lemma, we obtain the proposition given below.

Proposotion. *Let T be any contractive co-hyponormal unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathscr{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0\in\mathbb{D}$, then the operator T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

The proof of this Proposition yields a slightly stronger result for non-zero points in $\mathbb{D}$.

Lemma. *Let T be any contractive unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$, $w_0 \neq 0$, then the operator T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

Let T be a operator in $B_1(\mathbb{D})$. Let ζ be a fixed but arbitrary point in $\mathbb{D}$ and ϕ_ζ be an automorphism of the unit disc taking ζ to 0. The operator $\phi_\zeta(T)$ is also in $B_1(\mathbb{D})$ and may be realized as the adjoint of multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$. We may assume without loss of generality that the reproducing kernel $K_{\phi_\zeta(T)}$ of the Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$, is normalized at 0, that is, $K_{\phi_\zeta(T)}(z,0) = 1$ for all z in a neighbourhood of 0. In the following theorem it is shown that the question of R. G. Douglas has an affirmative answer in the class of co-subnormal contraction after imposing a mild assumption on the Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$.

Theorem. *Let T be a co-subnormal operator in $B_1(\mathbb{D})$ with $\|T\| \leq 1$. Let ζ be a fixed but arbitrary point in $\mathbb{D}$. Assume that polynomials are dense in $\mathcal{H}_{\phi_\zeta(T)}$ and that $\mathcal{K}_T(\zeta) = -\frac{1}{(1-|\zeta|^2)^2}$, then T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

Chapter 3 – Extremal operator and Uniqueness: The following lemma gives a natural bijective correspondence between the bundle shifts $\{T_{E_\alpha}$ on $(H^2_{E_\alpha}, d\omega_p) : \alpha \in \mathbb{T}^n\}$ and the bundle shifts $\{T_{E_\beta}$ on $(H^2_{E_\beta}, ds) : \beta \in \mathbb{T}^n\}$, which preserves the (respective) unitary equivalence class.

Lemma. *If v be a positive continuous function on $\partial\Omega$, then there exist a function F in $H^\infty_\gamma(\Omega)$ for some character γ such that $|F|^2 = v$ almost everywhere (w.r.t arc length measure) on $\partial\Omega$. In fact F is invertible in the sense that there exist G in $H^\infty_{\gamma^{-1}}(\Omega)$ so that $FG = 1$ on Ω.*

An alternative proof of the following characterization of pure, rationally cyclic, subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum in $\partial\Omega$, proved by Abrahamse and Douglas (see [2, Theorem 11]) have been obtained using the preceding lemma.

Theorem. *Every pure, rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and whose normal spectrum lies in $\partial\Omega$, is unitarily equivalent to a bundle shift T_{E_α} on $\left(H^2_{E_\alpha}, d\omega_p\right)$ for some character α.*

Let $(H^2(\Omega), \lambda ds)$ be the weighted Hardy space, where λ is some positive continuous function on $\partial\Omega$. Since the multiplication operator M on $(H^2(\Omega), \lambda ds)$ is a pure subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum equal to $\partial\Omega$, it must be unitarily equivalent to a bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$ for some character α. We determine the character α explicitly. Consequently, the lemma given below follows.

Lemma. *Let λ_1, λ_2 be two positive continuous function on $\partial\Omega$. Let u_{λ_k} be the harmonic function on Ω with continuous boundary value $\frac{1}{2}\log\lambda_k$. Then the operators M on the Hilbert spaces*

$\big(H^2(\Omega), \lambda_1 ds\big)$ and $\big(H^2(\Omega), \lambda_2 ds\big)$ are unitarily equivalent if and only if

$$\exp\big(ic_j(\lambda_1)\big) = \exp\big(ic_j(\lambda_2)\big), \quad j = 1, \dots, n,$$

where the constants $c_j(\lambda_k)$ are given by

$$c_j(\lambda_k) = -\int_{\partial\Omega_j} \frac{\partial}{\partial\eta_z}\big(u_{\lambda_k}(z)\big) ds(z), \quad \textit{for } j = 1, 2, ..., n, k = 1, 2.$$

It is also shown that given any character α, there exists a positive continuous function λ on $\partial\Omega$ so that the operator M on $(H^2(\Omega), \lambda ds)$ is unitarily equivalent to the bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$.

Then which of the operators M^* on the weighted Hardy space $(H^2(\Omega), \lambda ds)$ attain equality, in the curvature inequality at a fixed point $\bar{\zeta} \in \Omega^*$ are found. A criterion for the case of equality at one point is given. The Garabedian kernel $L_w(z)$ associated to the usual Hardy space $(H^2(\Omega), ds)$ together with the Sz̈ego kernel are significant in the study of the conformal geometry of a finitely connected domain Ω. Nehari has established the existence of a Garabedian like kernel $L^\lambda_w(z)$ associated to the Hardy space $(H^2(\Omega), \lambda ds)$, which is essential in describing the criterion for equality at a point in the curvature inequality as shown below.

Lemma. *The operator M^* on the Hilbert space $\big(H^2(\Omega), \lambda ds\big)$ is extremal at $\bar{\zeta}$ if and only if $L^{(\lambda)}_\zeta(z)$ and the Sz̈ego kernel at ζ, namely $S_\zeta(z)$ have the same set of zeros in Ω.*

Using the preceding lemma, a positive continuous function λ on $\partial\Omega$ for which M^* on the Hilbert space $(H^2(\Omega), \lambda ds)$ is extremal at $\bar{\zeta}$ is constructed. Next, the uniqueness of the extremal operator at $\bar{\zeta}$ in the class of “adjoint of the bundle shifts" is established.

Theorem (Uniqueness)**.** *Let ζ be an arbitrary but fixed point in Ω. If the bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$ and the bundle shift T_{E_β} on $(H^2_{E_\beta}, ds)$ are extremal at the point $\bar{\zeta}$, that is, if they satisfy*

$$\mathcal{K}_{T^*_{E_\alpha}}(\bar{\zeta}) = -4\pi^2 (S_{\Omega^*}(\bar{\zeta}, \bar{\zeta}))^2 = \mathcal{K}_{T^*_{E_\beta}}(\bar{\zeta}),$$

then the bundle shifts T_{E_α} and T_{E_β} are unitarily equivalent, which is the same as $\alpha = \beta$.

It is shown that the the character α for the bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$ which is extremal at $\bar{\zeta}$, is uniquely determined by the following n tuple of complex number of unit modulus:

$$\big\{\exp\big(2\pi i(1 - \omega_1(\zeta))\big), ..., \exp\big(2\pi i(1 - \omega_n(\zeta))\big)\big\},$$

where $\omega_j(z)$ is the harmonic function on Ω whose boundary value equal to 1 on $\partial\Omega_j$ and 0 on other boundary components.

From this result, it follows that if $n \geq 2$, then the set of extremal operators does not include the adjoint of many of the bundle shifts. When Ω is doubly connected, it turns out that for every character α, except one, there is a point $\bar{\zeta} \in \Omega^*$ such that the bundle shift T_{E_α} on $(H^2_{E_\alpha}, ds)$ is an extremal operator at the point $\bar{\zeta}$.

Chapter 4 – Generalized Curvature Inequality: In this chapter we consider a bounded domain Ω in $\mathbb{C}^m$. The matricial representation of the localization of a commuting m-tuple of bounded operators in the Cowen-Douglas class $B_n(\Omega^*)$, $\Omega^* \subseteq \mathbb{C}^m$, is given. Let $\boldsymbol{T} = (T_1, \ldots, T_m)$ be a commuting m-tuple of bounded operators in $B_n(\Omega^*)$ and $E_{\boldsymbol{T}}$ be the associated rank n holomorphic Hermitian vector bundle over Ω^* induced by the operator $\boldsymbol{T}$. Let $\mathcal{M}_w$ be the (finite dimensional) space $\cap_{i,j=1}^m \ker\big((T_i - w_i)(T_j - w_j)$, $w \in \Omega^*$ and $N_w = (N_1(w), \ldots, N_m(w))$ be the tuple of nilpotent operator defined by $N_i(w) = (T_i - w_i) \mid_{\mathcal{M}_w}$. We denote the block operator matrix $\big(\!\!\big(N_i(w) N_j(w)^*\big)\!\!\big)_{i,j=1}^m$ by $N_w N_w^*$.

Proposotion. *There exists an orthonormal basis in $\mathcal{M}_w$ such that the matrix representation of $N_w N_w^*$ with respect to this basis takes the form,*

$$N_w N_w^* = \begin{pmatrix} \mathcal{K}(w)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \; w \in \Omega^*,$$

where the curvature $\mathcal{K}$ of the bundle $E_{\boldsymbol{T}}$ is computed w.r.t a frame, defined on a neighborhood of w, which is orthonormal at w.

This representation makes it possible to use methods very similar to the ones employed in the case of operators in the case of $B_1(\Omega^*)$ to obtain a curvature inequality for commuting m- tuples $\boldsymbol{T} = (T_1, \ldots, T_m)$ in $B_n(\Omega^*)$, which admit Ω^* as a joint spectral set. In particular, for $m = 1$, for a fixed $w \in \Omega^*$, we obtain the inequality

$$\mathcal{K}(w) \leq -4\pi^2 S_{\Omega^*}(w, w)^2 I_n,$$

where the curvature $\mathcal{K}$ of the bundle E_T is computed with respect to a frame defined in a neighbourhood of w, which is orthonormal at w.

Chapter 5 – Module tensor product and dilation:

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\mathbb{C}^2_w(a)$ be the Hilbert module over the function algebra $\mathcal{O}(\overline{\Omega})$, where the module action is defined by the map:

$$(r, h) \mapsto \left(\begin{smallmatrix} r(w) & (\nabla r(w) \cdot a) \\ 0 & r(w) \end{smallmatrix}\right) h, \; a \in \mathbb{C}^m, w \in \Omega, \; r \in \mathcal{O}(\overline{\Omega}), h \in \mathbb{C}^2.$$

The module tensor product of $\mathcal{H}_K$ and $\mathbb{C}^2_w(a)$ over the function algebra $\mathcal{O}(\overline{\Omega})$, that is, $\mathcal{H}_K \otimes_{\mathcal{O}(\overline{\Omega})} \mathbb{C}^2_w(a)$ is shown to be isomorphic to $\mathbb{C}^2_w(\hat{a})$, where $\hat{a}$ is the vector given by

$$\hat{a} = \frac{a}{\sqrt{1 - \langle \mathcal{K}_K(\bar{w}) a, a \rangle}}, \; w \in \Omega.$$

In the case of a planar domain Ω, we prove the following lemma on the continuity of the curvature at a fixed point w in Ω as a function of the characters α.

Lemma. *For a fixed w in Ω, the function $\phi : \mathbb{T}^n \to \mathbb{R}$ defined by $\phi(\alpha) = \mathscr{K}_\alpha(\bar{w})$ is continuous.*

The existence of the extremal operators at w, along with the Lemma on continuity of the curvature are essential ingredients in the construction of an explicit dilation for the contractive modules of the form $\mathbb{C}^2_w(a)$ over the function algebra $\mathscr{O}(\overline{\Omega})$. In the case of the unit disc $\mathbb{D}$, it is well known that any two minimal Ŝilov resolution for a contractive module $\mathscr{M}$ over $\mathscr{O}(\overline{\mathbb{D}})$ are isomorphic. However this fails in the case of a finitely connected domain. Using the Lemma on continuity of curvature, we provide another example to demonstrate this phenomenon.

Chapter 2

Curvature Inequality and The case of the unit disc

2.1 Curvature Inequality

In the beginning, we assume that Ω is a bounded finitely connected planar domain whose boundary consists of $n+1$ jordan analytic curves. Setting $\Omega^* = \{\bar{z} : z \in \Omega\}$, we then describe the functional calculus for the local operators $T_{|\ker(T-w)^2}$ for T in $B_1(\Omega^*)$. As a consequence, we obtain a curvature inequality for the operator T which admits $\overline{\Omega}^*$ as a spectral set. We will then specialize to the case $\Omega = \mathbb{D}$.

Let T be an operator in $B_1(\Omega^*)$. We assume without loss of generality that the operator T has been realized as M^* on a Hilbert space $\mathscr{H}_K$ of holomorphc functions defined on Ω possessing a reproducing kernel K on $\Omega \times \Omega$. The curvature $\mathscr{K}_T(\bar{w})$ of the operator T can be computed using kernel function by following formula

$$\mathscr{K}_T(\bar{w}) = -\frac{\partial^2}{\partial z \partial \bar{z}} \log K(z,z)|_{z=w} = -\tfrac{\|K_w\|^2 \|\bar{\partial} K_w\|^2 - |\langle K_w, \bar{\partial} K_w\rangle|^2}{(K(w,w))^2}, \;\; \bar{w} \in \Omega^*,$$

Since $(M^* - \bar{w})K_w = 0$ for all $\bar{w} \in \Omega^*$, it follows that $(M^* - \bar{w})\bar{\partial} K_w = K_w$. The subspace $\ker(M^* - \bar{w})^2 = \mathrm{span}\{\mathrm{K_w}, \bar{\partial}\mathrm{K_w}\}$ of $\mathscr{H}_K$ is invariant for M^*. Representing the restriction of the operator M^* to this subspace with respect to the orthonormal basis, obtained from $\{K_w, \bar{\partial} K_w\}$ by applying Gram-Schmidt process, we have

$$M^*_{|\ker(M^* - \bar{w})^2} = \begin{pmatrix} \bar{w} & \frac{1}{\sqrt{-\mathscr{K}_T(\bar{w})}} \\ 0 & \bar{w} \end{pmatrix}.$$

For any $r \in \mathrm{Rat}(\overline{\Omega}^*)$, it is not hard to verify that

$$r\big(M^*_{|\ker(M^* - \bar{w})^2}\big) = \begin{pmatrix} r(\bar{w}) & \frac{r'(\bar{w})}{\sqrt{-\mathscr{K}_T(\bar{w})}} \\ 0 & r(\bar{w}) \end{pmatrix}.$$

Assume that $\overline{\Omega}^*$, the closure of Ω^*, is a spectral set for the operator T in $B_1(\Omega^*)$. It follows that $\overline{\Omega}^*$ is also a spectral set for $M^*\mid_{\ker(\mathrm{M}^*-\bar{\mathrm{w}})^2}$. It is well known that

$$\sup\{|r'(\bar{w})| : \|r\|_\infty \le 1, r(\bar{w}) = 0,\ r \in \mathrm{Rat}(\overline{\Omega}^*)\} = 2\pi(S_{\Omega^*}(\bar{w},\bar{w})) = (1-|w|^2)^{-1},\ \bar{w} \in \Omega^*,$$

where $S_{\Omega^*}(z,w)$ denotes the Szëgo kernel of Ω^* which is equal to the reproducing kernel for the Hardy space $(H^2(\Omega^*), ds)$ (cf. [7, Theorem 13.1]). Now the spectral set condition for $M^*\mid_{\ker(\mathrm{M}^*-\bar{\mathrm{w}})^2}$ will lead us to a curvature inequality (see [26, Corollary 1.2]), that is,

$$\mathcal{K}_T(\bar{w}) \le -4\pi^2(S_{\Omega^*}(\bar{w},\bar{w}))^2, \quad \bar{w} \in \Omega^*. \tag{2.1}$$

Equivalently, since $S_\Omega(z,w) = S_{\Omega^*}(\bar{w},\bar{z}),\ z,w \in \Omega$, the curvature inequality takes the form

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w,w) \ge 4\pi^2(S_\Omega(w,w))^2, \quad w \in \Omega. \tag{2.2}$$

This inequality we call as curvature inequality for an operator T in $B_1(\Omega^*)$ which possessed $\overline{\Omega}^*$ as a spectral set.

2.2 The case of unit disc

The standard unilateral backward shift operator U_+^* on $\ell^2(\mathbb{N})$ is defined by

$$U_+^*(a_0, a_1, a_2, \ldots) = (a_1, a_2, \ldots).$$

The operator $U_+^* \in B_1(\mathbb{D})$ and U_+^* is unitarily equivalent to the adjoint operator M^* on the Hardy space $(H^2(\mathbb{D}), ds)$. The reproducing kernel of the Hardy space, as is well-known, is the Szëgo kernel $S_{\mathbb{D}}(z,a)$ of the unit disc $\mathbb{D}$. It is given by the formula $S_{\mathbb{D}}(z,a) = \frac{1}{2\pi(1-z\bar{a})}$, for all z, a in $\mathbb{D}$. The computation of the curvature of the operator U_+^* is now straightforward and is given by the formula

$$-\mathcal{K}_{U_+^*}(w) = \frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\mathbb{D}}(w,w) = 4\pi^2(S_{\mathbb{D}}(w,w))^2, \quad w \in \mathbb{D}:$$

Since the closed unit disc is a spectral set for any contraction T (by Von Neumann inequality), it follows from equation (2.1) that the curvature of the operator U_+^* dominates the curvature of every other contraction T in $B_1(\mathbb{D})$.

$$\mathcal{K}_T(w) \le \mathcal{K}_{U_+^*}(w) = -(1-|w|^2)^{-2}, \ w \in \mathbb{D}$$

Since the curvature function $\mathcal{K}_T(w)$ on $\mathbb{D}$, is a complete invariant for the unitary equivalence class of T in $B_1(\mathbb{D})$, the following question of R. G. Douglas is a natural one.

Question 2.1 (R. G. Douglas). For a contraction T in $B_1(\mathbb{D})$, if $\mathscr{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some fixed w_0 in $\mathbb{D}$, then does it follow that T must be unitarily equivalent to the operator U_+^*?

It is known that the answer is negative (cf. [25]). However this question has an affirmative answer if, for instance, T is a homogeneous operator in $B_1(\mathbb{D})$ with $\|T\| \leq 1$ (cf. [25]). We find that $\mathscr{K}_T(\bar{\zeta}) = -(1-|\zeta|^2)^{-2}$ for some $\zeta \in \mathbb{D}$, if and only if the two vectors $\tilde{K}_\zeta$ and $\bar{\partial}\tilde{K}_\zeta$ are linearly dependent, where $\tilde{K}_w(z) = (1-z\bar{w})K_w(z)$ and $K_w(z)$ is the kernel function associated with the operator T in $B_1(\mathbb{D})$. This simple characterization shows that the question of Douglas has an affirmative answer in the class of contractive, co-hyponormal backward weighted shifts. Finally, using a criterion for subnormality due to Agler [3] and imposing a "mild assumption" on the operator, we obtain the same answer in the class of co-subnormal contractions in $B_1(\mathbb{D})$.

Let T be an operator in $B_1(\mathbb{D})$ and $\mathscr{H}_K$ be the associated reproducing kernel Hilbert space so that operator T has been realized as M^* on the Hilbert space $\mathscr{H}_K$. Without loss of generality we can assume $K_w \neq 0$ for every $w \in \mathbb{D}$. Let $w_1, \ldots, w_n$ be n arbitrary points in $\mathbb{D}$ and $c_1, \ldots, c_n$ be arbitrary complex numbers. Using the reproducing property of K and the property that $M^*(K_{w_i}) = \bar{w}_i K_{w_i}$ we will have

$$\|M^*(\sum_{i,j=1}^{n} c_i K_{w_i})\|^2 = \sum_{i,j=1}^{n} w_i \bar{w}_j K(w_i, w_j) c_j \bar{c}_i\ ,\ \|\sum_{i,j=1}^{n} c_i K_{w_i})\|^2 = (\sum_{i,j=1}^{n} K(w_i, w_j) c_j \bar{c}_i.$$

Let $\tilde{K}(z,w)$ be the function $(1-z\bar{w})K(z,w)$, $z, w \in \mathbb{D}$. Now it is easy to see that the operator M^* on the Hilbert space $\mathscr{H}_K$ is contractive if and only if $\tilde{K}$ is non-negative definite.

Lemma 2.2. *Let T be a contraction in $B_1(\mathbb{D})$ and $\mathscr{H}_K$ be the associated reproducing kernel Hilbert space. Then for an arbitrary but fixed $\zeta \in \mathbb{D}$, we have $\mathscr{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ if and only if the vectors $\tilde{K}_\zeta, \bar{\partial}\tilde{K}_\zeta$ are linearly dependent in the Hilbert space $\mathscr{H}_{\tilde{K}}$.*

Proof. Assume $\mathscr{K}_{M^*}(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ for some $\zeta \in \mathbb{D}$. Contractivity of M^* gives us the function $\tilde{K} : \mathbb{D} \times \mathbb{D} \mapsto \mathbb{C}$ defined by

$$\tilde{K}(z,w) = (1-z\bar{w})K(z,w) \quad z, w \in \mathbb{D}$$

is a non negative definite kernel function. Consequently there exist a reproducing kernel Hilbert space $\tilde{\mathscr{H}}$, consisting of complex valued function on $\mathbb{D}$ such that $\tilde{K}$ becomes the reproducing kernel for $\tilde{\mathscr{H}}$. Also note that $\tilde{K}(z,z) = (1-|z|^2)K(z,z) \neq 0$, for $z \in \mathbb{D}$ which gives us $\tilde{K}_z \neq 0$. Let ζ be an arbitrary but fixed point in $\mathbb{D}$. Now it is straightforward to verify that $\mathscr{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ if and only if $\frac{\partial^2}{\partial z \bar{\partial} z} \log \tilde{K}(z,z)|_{z=\zeta} = 0$. Since we have

$$\frac{\partial^2}{\partial z \bar{\partial} z} \log \tilde{K}(z,z)|_{z=\zeta} = -\frac{\|\tilde{K}_\zeta\|^2 \|\bar{\partial}\tilde{K}_\zeta\|^2 - |\langle \tilde{K}_\zeta, \bar{\partial}\tilde{K}_\zeta \rangle|^2}{(\tilde{K}(\zeta,\zeta))^2},$$

Using Cauchy-Schwarz inequality, we see that the proof is complete. $\square$

Remark 2.3. Two non-zero linear functional g_1, g_2 on a vector space are linearly dependent if and only if $\ker(g_1) = \ker(g_2)$. Since $\tilde{K}_\zeta \neq 0$, there are two different possibilities for the linear dependence of the two vectors $\tilde{K}_\zeta, \bar{\partial}\tilde{K}_\zeta$. First, $\bar{\partial}\tilde{K}_\zeta \equiv 0$, that is, $f'(\zeta) = \langle f, \bar{\partial}\tilde{K}_\zeta \rangle = 0$ for all $f \in \tilde{\mathscr{H}}$. Second, $\ker \bar{\partial}\tilde{K}_\zeta = \ker \tilde{K}_\zeta$, that is, the set $\{f \in \tilde{\mathscr{H}} \mid f'(\zeta) = 0\}$ is equal to the set $\{f \in \tilde{\mathscr{H}} \mid f(\zeta) = 0\}$

Remark 2.4. Let $e(w) = \frac{1}{\sqrt{2}}(\tilde{K}_w \otimes \bar{\partial}\tilde{K}_w - \bar{\partial}\tilde{K}_w \otimes \tilde{K}_w)$ for $w \in \mathbb{D}$. A straightforward computation shows that $\|e(w)\|^2_{\tilde{\mathscr{H}}\otimes\tilde{\mathscr{H}}} = \tilde{K}(w,w)^2 \frac{\partial^2}{\partial z \bar{\partial} z} \log \tilde{K}(z,z)|_{z=w}$. Now if we define

$$F_T(z,w) := \langle e(z), e(w) \rangle_{\tilde{\mathscr{H}}\otimes\tilde{\mathscr{H}}} \text{ for } z, w \in \mathbb{D},$$

then clearly F_T is a non negative definite kernel function on $\mathbb{D}\times\mathbb{D}$. In view of this, we conclude that $\mathscr{K}_T(\bar{\zeta}) = -(1-|\zeta|^2)^{-2}$ if and only if $F_T(\zeta,\zeta) = 0$.

Proposition 2.5. *Let T be any contractive co-hyponormal unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathscr{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$, then the operator T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

Proof. Let T be a contraction in $B_1(\mathbb{D})$ and $\mathscr{H}_K$ be the associated reproducing kernel Hilbert space so that T is unitarily equivalent to the operator M^* on $\mathscr{H}_K$. By our hypothesis on T we have that operator M on $\mathscr{H}_K$ is a unilateral forward weighted shift. Without loss of generality, we may assume that the reproducing kernel K is of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n z^n \bar{w}^n, \;\; z, w \in \mathbb{D}; \text{ where } a_n > 0 \text{ for all } n \geq 0.$$

By our hypothesis on the operator T, we have that the operator M on $\mathscr{H}_K$ is a contraction. So, the function $\tilde{K}$ defined by $\tilde{K}(z,w) = (1-z\bar{w})K(z,w)$ is a non negative definite kernel function. Consequently, following the Remark 2.4, the function $F_T(w,w)$ defined by $F_T(w,w) = \tilde{K}(w,w)^2 \frac{\partial^2}{\partial z \bar{\partial} z} \log \tilde{K}(z,z)|_{z=w}$ is also non negative definite. The kernel $K(w,w)$ is a weighted sum of monomials $z^k \bar{w}^k$, $k = 0,1,2,\ldots$. Hence both $\tilde{K}(w,w)$ and $F_T(w,w)$ are also weighted sums of the same form. So, we have

$$F_T(w,w) = \sum_{n=0}^{\infty} c_n |w|^{2n},$$

for some $c_n \geq 0$. Now assume $\mathscr{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$ for some ζ in $\mathbb{D}$.

Case 1: If $\zeta \neq 0$, then following Remark 2.4, we have

$$F_T(\zeta,\zeta) = \sum_{n=0}^{\infty} c_n |\zeta|^{2n} = 0.$$

Thus $c_n = 0$ for all $n \geq 0$ since $c_n \geq 0$ and $|\zeta| \neq 0$. It follows that F_T is identically zero on $\mathbb{D}\times\mathbb{D}$, that is, $\frac{\partial^2}{\partial z \bar{\partial} z} \log \tilde{K}(z,z)|_{z=\bar{w}} = 0$ for all $w \in \mathbb{D}$. Hence

$$\frac{\partial^2}{\partial z \bar{\partial} z} \log K(z,z)|_{z=\bar{w}} = \frac{\partial^2}{\partial z \bar{\partial} z} \log S_{\mathbb{D}}(z,z)|_{z=\bar{w}} \text{ for all } w \in \mathbb{D}.$$

Therefore, $\mathscr{K}_T(\bar{w}) = \mathscr{K}_{U_+^*}(\bar{w})$ for all $w \in \mathbb{D}$ making $T \cong U_+^*$.

Now let's discuss the remaining case, that is $\mathscr{K}_T(\bar{\zeta}) = -\frac{1}{(1-|\zeta|^2)^2}$, for $\zeta = 0 \in \mathbb{D}$

Case 2: If $\zeta = 0$, then by Lemma 2.2, we have $\tilde{K}_0, \bar{\partial}\tilde{K}_0$ are linearly dependent. Now,

$$\tilde{K}(z,w) := (1 - z\bar{w})K(z,w) = \sum_{n=0}^{\infty} b_n z^n \bar{w}^n,$$

where $b_0 = a_0$ and $b_n = a_n - a_{n-1} \geq 0$, for all $n \geq 1$. Consequently, we have $\tilde{K}_0(z) \equiv b_0$ and $\bar{\partial}\tilde{K}_0(z) = b_1 z$. Now $\tilde{K}_0, \bar{\partial}\tilde{K}_0$ are linearly dependent if and only if $b_1 = 0$ that is $a_0 = a_1$.

As an aside, we note that using this criterion, it is easy to construct a family of contractive unilateral backward weighted shift operator T, not unitarily equivalent to the usual backward shift operator U_+^*, in $B_1(\mathbb{D})$ for which $\mathscr{K}_T(0) = -1$.

Since $\{\sqrt{a_n} z^n\}_{n=0}^{\infty}$ is an orthonormal basis for the Hilbert space $\mathscr{H}_K$, the operator M on $\mathscr{H}_K$ is an unilateral forward weighted shift with weight sequence $w_n = \sqrt{\frac{a_n}{a_{n+1}}}$ for $n \geq 0$. So the curvature of M^* at the point zero equal to -1 if and only if $w_0 = \sqrt{\frac{a_0}{a_1}} = 1$. Now if we further assume M is hyponormal, that is, $M^*M \geq MM^*$, then the sequence w_n must be increasing. Also contractivity of M implies that $w_n \leq 1$. Therefore if $\mathscr{K}_{M^*}(0) = -1$ for some contractive hyponormal backward weighted shift M^* in $B_1(\mathbb{D})$, then it follows that $w_n = 1$ for all $n \geq 1$. Thus any such operator is unitarily equivalent to the backward unilateral shift U_+^* completing the proof of our claim. □

The proof of **Case 1** given above proves a little more than what is stated in the proposition, which we record below as a separate Lemma.

Lemma 2.6. *Let T be any contractive unilateral backward weighted shift operator in $B_1(\mathbb{D})$. If $\mathscr{K}_T(w_0) = -(1 - |w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$, $w_0 \neq 0$, then the operator T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

Let T be a operator in $B_1(\mathbb{D})$. Let ζ be a fixed but arbitrary point in $\mathbb{D}$ and ϕ_ζ be an automorphism of the unit disc taking ζ to 0. So, we have $\phi_\zeta(z) = \alpha \frac{z-\zeta}{1-\bar{\zeta}z}$ for some unimodular constant α. Note that

$$\phi_\zeta(T) - w = \alpha(T - \zeta)(1 - \bar{\zeta}T)^{-1} - w = \alpha(T - \tfrac{\bar{\alpha}w+\zeta}{1+\bar{\zeta}\bar{\alpha}w})(1 + \bar{\zeta}\bar{\alpha}w)(1 - \bar{\zeta}T)^{-1},\ w \in \mathbb{D}.$$

From there it follows that $\phi_\zeta(T)$ is also in $B_1(\mathbb{D})$. Let $\gamma(w)$ be a frame for the associated bundle E_T of T so that $T(\gamma(w)) = w\gamma(w)$ for all $w \in \mathbb{D}$. Now it is easy to see that $\phi_\zeta(T)(\gamma(w)) = \phi_\zeta(w)\gamma(w)$ or equivalently $\phi_\zeta(T)(\gamma \circ \phi_\zeta^{-1}(w)) = w(\gamma \circ \phi_\zeta^{-1}(w))$. So, $\gamma \circ \phi_\zeta^{-1}(w)$ is a frame for the bundle $E_{\phi_\zeta(T)}$ associated with $\phi_\zeta(T)$. Hence the curvature $\mathscr{K}_{\phi_\zeta(T)}(w)$ is equal to

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log\|\gamma \circ \phi_\zeta^{-1}(w)\|^2 = |\phi_\zeta^{-1\prime}(w)|^2 \frac{\partial^2}{\partial z \partial \bar{z}} \log\|\gamma(z)\|^2_{|z=\phi_\zeta^{-1}(w)} = |\phi_\zeta^{-1\prime}(w)|^2 \mathscr{K}_T(\phi_\zeta^{-1}(w)).$$

This leads us to the following transformation rule for the curvature

$$\mathcal{K}_{\phi_\zeta(T)}(\phi_\zeta(z)) = \mathcal{K}_T(z)|\phi_\zeta'(z)|^{-2},\ z \in \mathbb{D}. \tag{2.3}$$

Since $|\phi_\zeta'(\zeta)| = (1-|\zeta|^2)^{-1}$, in particular we have that

$$\mathcal{K}_{\phi_\zeta(T)}(0) = \mathcal{K}_T(\zeta)(1-|\zeta|^2)^2. \tag{2.4}$$

Since $\phi_\zeta(T) \in B_1(\mathbb{D})$, it may be realized as the adjoint of multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$. We may assume without loss of generality that the reproducing kernel $K_{\phi_\zeta(T)}$, associated to the Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$, is normalized at 0, that is, $K_{\phi_\zeta(T)}(z,0) = 1$ for all z in a neighbourhood of 0. In the following theorem it is shown that the question of R. G. Douglas has an affirmative answer in the class of co-subnormal contraction with a mild assumption on the Hilbert space $\mathcal{H}_{\phi_\zeta(T)}$.

Theorem 2.7. *Let T be a co-subnormal operator in $B_1(\mathbb{D})$ with $\|T\| \le 1$. Let ζ be a fixed but arbitrary point in $\mathbb{D}$. Assume that polynomials are dense in $\mathcal{H}_{\phi_\zeta(T)}$ and that $\mathcal{K}_T(\zeta) = -\frac{1}{(1-|\zeta|^2)^2}$, then T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator.*

Proof. Let T be a co-subnormal operator in $B_1(\mathbb{D})$ with $\|T\| \le 1$. Let P be the operator $\phi_\zeta(T)$. It is straightforward to see that P is again a co-subnormal operator in $B_1(\mathbb{D})$ with $\|P\| \le 1$. Now assume $\mathcal{K}_T(\zeta) = -(1-|\zeta|^2)^{-2}$. Following (2.4), we get that $\mathcal{K}_P(0) = -1$.

Let $\mathcal{H}_K$ be the associated reproducing kernel Hilbert space with kernel function K so that P is unitarily equivalent to M^* on $\mathcal{H}_K$. Without loss of generality, we can assume that the kernel function K is normalized at 0, that is, $K_0(z) = K(z,0) = 1$ for all z in some neighbourhood of 0. By assumption, we then have polynomials are dense in $\mathcal{H}_K$. Since the kernel K is analytic in first variable and co-analytic in second variable, let $K(z,w) = \sum_{m,n\ge 0} a_{m,n} z^m \bar{w}^n$, for all z, w in some neighborhood of zero. As $K(z,0) = K(0,z) = 1$, we get that $a_{i,0} = a_{0,i} = 0$, for all $i \ge 1$ and $a_{0,0} = 1$. Note that since K is normalized at 0, we have $\mathcal{K}_P(0) = -\bar\partial\partial K(0,0) = -1$.

Consider the kernel function $\tilde{K}(z,w) = (1-z\bar{w})K(z,w)$, for $z, w \in \mathbb{D}$. From Lemma 2.2, it follows that $\tilde{K}_0(z)$ and $\bar\partial\tilde{K}_0(z)$ are linearly dependent. Since $\tilde{K}_0(z) = K_0(z) \equiv 1$, we must have $\bar\partial\tilde{K}_0(z) \equiv c$ for some scalar c.

$$\tilde{K}(z,w) = 1 + \sum_{m,n\ge 0}(a_{m+1,n+1} - a_{m,n}) z^{m+1}\bar{w}^{n+1},$$
$$\bar\partial\tilde{K}_0(z) = \sum_{m\ge 0}(a_{m+1,1} - a_{m,0}) z^{m+1} = (a_{1,1} - a_{0,0})z + \sum_{m\ge 1} a_{m+1,1} z^{m+1}.$$

Consequently, we have that $c = 0$, $a_{1,1} = a_{0,0} = 1$ and $a_{i,1} = 0$, for $i \ge 2$. Hence, we get that $\bar\partial K_0(z) = z$. By hypothesis, we also have $\mathcal{K}_P(0) = -\bar\partial\partial K(0,0) = -1$. So, we have

$$\|z\|^2_{\mathcal{H}_K} = \|\bar\partial K_0(z)\|^2 = \bar\partial\partial K(0,0) = 1 \text{ and } \|1\|^2_{\mathcal{H}_K} = \|K_0(z)\|^2 = K(0,0) = 1.$$

As the operator M on $\mathscr{H}_K$ is contraction and $\|1\|_{\mathscr{H}_K} = 1$, we have $\|z^n\|_{\mathscr{H}_K} \leq 1$, for all $n \geq 1$. Since M on $\mathscr{H}_K$ is a contractive subnormal operator, following Agler [3, Theorem 3.1], we have that M is n-hyper contraction for every $n \in \mathbb{N}$. In particular, M is 2-hyper contraction, that is, $I - 2M^*M + M^{*2}M^2 \geq 0$, equivalently, $\|f\|^2_{\mathscr{H}_K} - 2\|zf\|^2_{\mathscr{H}_K} + \|z^2 f\|^2_{\mathscr{H}_K} \geq 0$, for all $f \in \mathscr{H}_K$. Since $\|1\| = \|z\| = 1$, taking $f = 1$, we have $\|z^2\| \geq 1$. But we also have $\|z^2\| \leq 1$, which gives us $\|z^2\| = 1$. Inductively, by choosing $f = z^k$, we obtain $\|z^{k+2}\| = 1$ for every $k \in \mathbb{N}$. Hence we see that $\|z^n\| = 1$ for all $n \geq 0$.

To establish that $\{z^n \mid n \geq 0\}$ is orthonormal set in the Hilbert space $\mathscr{H}_K$, we need the Lemma given below.

Lemma 2.8. *Let V and W be two finite dimensional inner product space and $A : V \to W$ be a linear map. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for V and G_v, (resp. G_{Av}) be the grammian $(\!(\langle v_j, v_i\rangle_V)\!)$ (resp. $(\!(\langle Av_j, Av_i\rangle_W)\!)$). The linear map A is a contraction if and only if $G_{Av} \leq G_v$.*

Proof. Let $x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ be an arbitrary element in V. Then the easy verification that $\|Ax\|^2_W \leq \|x\|^2_V$ is equivalent $\langle G_{Av} c, c\rangle \leq \langle G_v c, c\rangle$ completes the proof. □

Consider the two subspace V and W of $\mathscr{H}_K$, defined by $V = \vee\{1, z, \cdots, z^k\}$ and $W = \vee\{z, z^2, \cdots, z^{k+1}\}$. Since M is a contraction, applying the lemma we have just proved, it follows that the matrix B defined by

$$B = \left(\langle z^j, z^i\rangle\right)_{i,j=0}^{k} - \left(\langle z^{j+1}, z^{i+1}\rangle\right)_{i,j=0}^{k}$$

is positive semi-definite. But we have $\|z^i\| = 1$, for all $i \geq 0$. Consequently, each diagonal entry of B is zero. Hence $tr(B) = 0$. Since B is positive semi-definite, it follows that $B = 0$. Therefore, $\langle z^j, z^i\rangle = \langle z^{j+1}, z^{i+1}\rangle$ for all $0 \leq i, j \leq k$. We have $K_0(z) \equiv 1$. So, $M^*1 = M^*(K_0) = 0$. From this it follows that for any $k \geq 1$, we have $\langle z^k, 1\rangle = \langle z^{k-1}, M^*1\rangle = 0$. This together with $\langle z^j, z^i\rangle = \langle z^{j+1}, z^{i+1}\rangle$ for all $0 \leq i, j \leq k$, inductively shows that $\langle z^j, z^i\rangle = 0$ for every $i \neq j$. Hence $\{z^n \mid n \geq 0\}$ forms an orthonormal set.

Since by hypothesis polynomials are dense in $\mathscr{H}_K$, we have $\{z^n \mid n \geq 0\}$ forms an orthonormal basis for $\mathscr{H}_K$. Hence the multiplication operator M on $\mathscr{H}_K$ is unitarily equivalent to U_+, the standard unilateral forward shift operator. Consequently P is unitarily equivalent to U_+^*. But U_+^* being a homogeneous operator, we have U_+^* is unitarily equivalent to $\phi_\zeta^{-1}(U_+^*)$ (cf. [25]). Hence, we get that $T = \phi_\zeta^{-1}(P)$ is unitarily equivalent to U_+^*.

□

Now we turn our attention from $\mathbb{D}$ to other finitely connected bounded planar domains Ω whose boundary consists of $n+1$ Jordan analytic curves. We have seen that if Ω is the unit disc, then $\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\mathbb{D}}(w,w) = 4\pi^2 (S_{\mathbb{D}}(w,w))^2, \ \ w \in \mathbb{D}$. On the other hand, for any bounded simply connected region Ω, the Sžego kernel $S_\Omega(z,w)$ is given by

$$S_\Omega(z,w) = \sqrt{F'(z)}\, S_{\mathbb{D}}(F(z), F(w)) \overline{\sqrt{F'(w)}},$$

where $F : \Omega \mapsto \mathbb{D}$ is any bi holomorphic map (cf. [7, Theorem 12.3]). Now using the Riemann map and the transformation rules for the curvature (2.3), we conclude that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_\Omega(w,w) = 4\pi^2 (S_\Omega(w,w))^2, \ \ w \in \Omega. \tag{2.5}$$

This shows that in the case of bounded simply connected domain with Jordan analytic boundary, the operator M^* on $(H^2(\Omega), ds)$ is an extremal operator, that is, $\mathcal{K}_T(\bar{w}) \le \mathcal{K}_{M^*}(\bar{w})$ for every operator $T \in B_1(\Omega^*)$ admitting Ω^* as a spectral set.

On the other hand, if the region is not simply connected, then (2.5) fails. Indeed, for such region, Suita (cf. [36]) has shown that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_\Omega(w,w) > 4\pi^2 (S_\Omega(w,w))^2, \ \ w \in \Omega. \tag{2.6}$$

Or equivalently,

$$\mathcal{K}_{M^*}(z) < -4\pi^2 (S_{\Omega^*}(z,z))^2, \ \ z \in \Omega^*, \tag{2.7}$$

where M^* is the adjoint of the multiplication by the coordinate function on the Hardy space $(H^2(\Omega), ds)$. We therefore conclude that if Ω is not simply connected, then the operator M^* on the Hardy space $(H^2(\Omega), ds)$ fails to be extremal.

We don't know if there exists an operator T in $B_1(\Omega^*)$ admitting $\overline{\Omega}^*$ as a spectral set for which $\mathcal{K}_T(w) = -4\pi^2 (S_{\Omega^*}(w,w))^2$, for all $w \in \Omega^*$. The question of equality at just one fixed but arbitrary point $\bar{\zeta}$ in Ω^* was answered in [26, Theorem 2.1].

Definition 2.9 (Extremal operator at point)**.** An operator T in $B_1(\Omega^*)$ for which $\overline{\Omega}^*$ is a spectral set is called extremal at $\bar{\zeta}$ if $\mathcal{K}_T(\bar{\zeta}) = -4\pi^2 (S_{\Omega^*}(\bar{\zeta}, \bar{\zeta}))^2$.

Representing the extremal operator T as the operator M^* on a Hilbert space possessing a reproducing kernel $K_T : \Omega \times \Omega \to \mathbb{C}$, we have that the operator T is an extremal at $\bar{\zeta}$ if and only if

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K_T(w,w)|_{w=\zeta} = 4\pi^2 S_\Omega(\zeta,\zeta)^2. \tag{2.8}$$

The operator M on the Hardy space $(H^2(\mathbb{D}), ds)$ is a pure subnormal operator with the property: the spectrum of the minimal normal extension, designated the normal spectrum, is contained in the boundary of the spectrum of the operator M. These properties determine the operator M uniquely up to unitary equivalence in the class of subnormal contractions. The question of characterizing all pure subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum contained in the boundary of Ω is more challenging if Ω is not simply connected. The deep results of Abrahamse and Douglas (cf. [2, Theorem 11]) show that these are exactly the bundle shifts, what is more, they are in one to one correspondence with the equivalence classes of flat unitary bundles on the domain Ω. It follows that adjoint of a bundle shift of rank 1 lies in $B_1(\Omega^*)$. Since bundle shifts are subnormal with spectrum equal to $\overline{\Omega}$, it follows that $\overline{\Omega}^*$ is a spectral set for the adjoint of the bundle shift. In fact, the extremal operator at $\bar{\zeta}$, found in [26], is the adjoint of a bundle shift of rank 1. Therefore, one may ask, following R. G. Douglas, if the curvature $\mathscr{K}_T(\bar{\zeta})$ of an operator T in $B_1(\Omega^*)$, admitting $\overline{\Omega}^*$ as a spectral set, equals $-4\pi^2 S_\Omega(\zeta,\zeta)^2$, then does it follow that T is necessarily unitarily equivalent to the extremal operator at $\bar{\zeta}$ found in [26]? In the following chapter, we show that an extremal operator must be uniquely determined within $\big\{[\![T^*]\!] : T \text{ is bundle shift of multiplicity 1 over } \Omega\big\}$, where $[\![\cdot]\!]$ denotes the unitary equivalence class.

Chapter 3

Extremal operator and Uniqueness

3.1 Hardy space and bundle shift

Let Ω be a bounded, open and connected subset of the complex plane $\mathbb{C}$. Assume that $\partial\Omega$, the boundary of Ω, consists of $n+1$ analytic Jordan curves. Let $\partial\Omega_1, \partial\Omega_2, \cdots, \partial\Omega_{n+1}$ denote the boundary components of Ω. We shall always let $\partial\Omega_{n+1}$ denote the curve whose interior contains Ω. Set $\Omega^* = \{\bar{z} \mid z \in \Omega\}$, which is again a planar domain whose boundary consists of $n+1$ analytic Jordan curves. Let p be a fixed but arbitrary point in Ω.

Let E_α be a flat unitary vector bundle over Ω induced by a character α in $\mathrm{Hom}(\pi_1(\Omega), \mathbb{T})$. Bundle shift T_{E_α} on the Hilbert space $(H^2_{E_\alpha}, d\omega_p)$ can also be realized as a multiplication operator on a certain subspace of the classical Hardy space $H^2(\mathbb{D})$. Let $\pi : \mathbb{D} \mapsto \Omega$ be a holomorphic covering map satisfying $\pi(0) = p$. Let G denote the group of deck transformation associated to the map π that is $G = \{A \in \mathrm{Aut}(\mathbb{D}) \mid \pi \circ \mathrm{A} = \pi\}$. As G is isomorphic to the fundamental group $\pi_1(\Omega)$ of Ω, every character α induce a unique element in $\mathrm{Hom(G}, \mathbb{T})$. By an abuse of notation we will also denote it by α. A holomorphic function f on unit disc $\mathbb{D}$ satisfying $f \circ A = \alpha(A) f, \text{ for all } A \in G$, is called a modulus automorphic function of index α. Now consider the following subspace of the Hardy space $H^2(\mathbb{D})$ which consists of modulus automorphic function of index α, namely

$$H^2(\mathbb{D}, \alpha) = \{f \in H^2(\mathbb{D}) \mid f \circ A = \alpha(A) f, \text{ for all } A \in G\}.$$

Let T_α be the multiplication operator by the covering map π on the subspace $H^2(\mathbb{D}, \alpha)$. Abrahamse and Douglas have shown in [2, Theorem 5] that the operator T_α is unitarily equivalent to the bundle shift T_{E_α} on $(H^2_{E_\alpha}, d\omega_p)$.

There is another way to realize the bundle shift as a multiplication operator M on a Hilbert space of multivalued holomorphic function defined on Ω with the property that its absolute value is single valued. A multivalued holomorphic function defined on Ω with the property that its absolute value is single valued is called a multiplicative function. Every modulus automorphic function f on $\mathbb{D}$ induce a multiplicative function on Ω, namely, $f \circ \pi^{-1}$. Converse is also true (see [40, Lemma 3.6]). We define the class $H^2_\alpha(\Omega)$ consisting of multiplicative function in the following way:

$$H^2_\alpha(\Omega) := \{f \circ \pi^{-1} \mid f \in H^2(\mathbb{D}, \alpha)\}.$$

So the linear space $H^2_\alpha(\Omega)$ is consisting of those multiple valued function h on Ω for which $|h|$ is single valued, $|h|^2$ has a harmonic majorant on Ω and h is locally holomorphic in the sense that each point $w \in \Omega$ has a neighbourhood U_w and a single valued holomorphic function g_w on U_w with the property $|g_w| = |h|$ on U_w (cf. [20, p.101]). Each function f in $H^2_\alpha(\Omega)$ admits a boundary value $\hat{f}$ that is the $\lim_{z\to\lambda} f(z)$ exists (almost everywhere relative to $d\omega_p$) as z approaches $\lambda \in \partial\Omega$ through any non-tangential path in Ω. The map $f \mapsto \hat{f}$ is an isomorphism between the linear spaces $H^2_\alpha(\Omega)$ and its boundary values. Because of this correspondence, for a function $f \in H^2_\alpha(\Omega)$, we will use the same symbol f to denote both the function f and its boundary value $\hat{f}$.

Since the covering map π lifts the harmonic measure $d\omega_p$ on $\partial\Omega$ at the point $\pi(0) = p$ to the linear Lebesgue measure on the unit circle $\mathbb{T}$, It follows that $H^2_\alpha(\Omega)$ endowed with the norm

$$\|f\|^2 = \int_{\partial\Omega} |f(z)|^2 d\omega_p(z),$$

becomes a Hilbert space (cf. [20, p. 101].) We will denote it by $\left(H^2_\alpha(\Omega), d\omega_p\right)$. In fact the map $f \mapsto f \circ \pi^{-1}$ is a unitary map from $H^2(\mathbb{D}, \alpha)$ onto $\left(H^2_\alpha(\Omega), d\omega_p\right)$ which intertwine the multiplication by π on $H^2(\mathbb{D}, \alpha)$ and the multiplication by coordinate function M on $\left(H^2_\alpha(\Omega), d\omega_p\right)$.

It is well known that the harmonic measure $d\omega_p$ on $\partial\Omega$ at the point p is boundedly mutually absolutely continuous w.r.t the arc length measure ds on $\partial\Omega$. In fact we have

$$d\omega_p(z) = -\frac{1}{2\pi}\frac{\partial}{\partial\eta_z}\big(g(z,p)\big)ds(z), \;\; z \in \partial\Omega,$$

where $g(z,\zeta)$ denote the green function for the domain Ω at the point p and $\frac{\partial}{\partial\eta_z}$ denote the directional derivative along the outward normal direction (w.r.t positively oriented $\partial\Omega$). In this paper, instead of working with harmonic measure $d\omega_p$ on $\partial\Omega$, we will work with arclength measure ds on $\partial\Omega$. This is the approach in Sarason [34]. So, we define the norm of a function f in in $H^2_\alpha(\Omega)$ by

$$\|f\|^2_{ds} = \int_{\partial\Omega} |f(z)|^2 ds.$$

Since the outward normal derivative of the Green's function is negative on the boundary $\partial\Omega$, we have

$$d\omega_p(z) = h^2(z)ds(z), \;\; z \in \partial\Omega, \tag{3.1}$$

where $h(z)$ is a positive continuous function on $\partial\Omega$. We also see that

$$c_2\|f\|_{ds}^2 \le \|f\|^2 \le c_1\|f\|_{ds}^2,$$

where c_1 and c_2 are the supremum and the infimum of the function h on $\partial\Omega$.

Hence it is clear that $\|\cdot\|_{ds}$ defines an equivalent norm on $H^2_\alpha(\Omega)$, We let $\left(H^2_\alpha(\Omega), ds\right)$ be the Hilbert space which is the same as $H^2_\alpha(\Omega)$ as a linear space but is given the new norm $\|\cdot\|_{ds}$. In fact, the identity map from $\left(H^2_\alpha(\Omega), d\omega_p\right)$ onto $\left(H^2_\alpha(\Omega), ds\right)$ is invertible and intertwines the corresponding multiplication operator by the coordinate function. It is easily verified that the multiplication operator by coordinate function on $\left(H^2_\alpha(\Omega), ds\right)$ is also a pure, rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum contained in $\partial\Omega$. By a slight abuse of notation, we will denote the multiplication operator by the coordinate function on $\left(H^2_\alpha(\Omega), ds\right)$ also by T_α.

Using the characterization of all rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum contained in $\partial\Omega$ given by Abrhamse and Douglas, we conclude that for every character β, the operator T_β on $\left(H^2_\beta(\Omega), ds\right)$ is unitarily equivalent to T_α on $\left(H^2_\alpha(\Omega), d\omega_p\right)$ for some α. In the following section we will establish a bijective correspondence (which respects the unitary equivalence class) between these two kinds of bundle shifts. The following Lemma helps in establishing this bijection.

Lemma 3.1. *If v be a positive continuous function on $\partial\Omega$, then there exist a function F in $H^\infty_\gamma(\Omega)$ for some character γ such that $|F|^2 = v$ almost everywhere (w.r.t arc length measure), on $\partial\Omega$. In fact F is invertible in the sense that there exist G in $H^\infty_{\gamma^{-1}}(\Omega)$ so that $FG = 1$ on Ω.*

Proof. Since v is a positive continuous function on $\partial\Omega$, it follows that $\log v$ is continuous on $\partial\Omega$. Since the boundary $\partial\Omega$ of Ω consists of jordan analytic curves, the Dirichlet problem is solvable with continuous boundary data. Now solving the Dirichlet problem with boundary value $\frac{1}{2}\log v$, we get a harmonic function u on Ω with continuous boundary value $\frac{1}{2}\log v$. Let u^* be the multiple value conjugate harmonic function of u. Let's denote the period of the multiple valued conjugate harmonic function u^* around the boundary component $\partial\Omega_j$ by

$$c_j = -\int_{\partial\Omega_j} \frac{\partial}{\partial\eta_z}\big(u(z)\big)ds_z, \;\; \text{for } j = 1,2,...,n.$$

In the above equation negative sign appear since we have assumed that $\partial\Omega$ is positively oriented, hence the different components of the boundary $\partial\Omega_j,\ j = 1,2,\ldots,n$, except the outer one are oriented in clockwise direction. Now consider the function $F(z)$ defined by

$$F(z) = \exp(u(z) + iu^*(z)).$$

Observe that F is a multiplicative holomorphic function on Ω. Hence following [40, Lemma 3.6], we have a existence of modulus automorphic function f on unit disc $\mathbb{D}$ so that $F = f \circ \pi^{-1}$. We find the index of the modulus automorphy for the function f in the following way. Around each boundary component $\partial\Omega_j$, along the anticlockwise direction, the value of F gets changed by $\exp(ic_j)$ times its initial value. So, the index of f is determined by the n tuple of numbers $(\gamma_1, \gamma_2, ..., \gamma_n)$ given by,

$$\gamma_j = \exp(ic_j), \quad j = 1, 2, ..., n.$$

For each of these n tuple of numbers, there exist a homomorphism $\gamma : \pi_1(\Omega) \to \mathbb{T}$ such that these n tuple of numbers occur as a image of the n generator of the group $\pi_1(\Omega)$ under the map γ. Also we have $|F(z)|^2 = \exp(2u(z)) = v(z), \quad z \in \partial\Omega$. Since u is continuous on $\overline{\Omega}$, it follows that $|F(z)|$ is bounded on Ω. Hence F belongs to $H^\infty_\gamma(\Omega)$ with $|F|^2 = v$ on $\partial\Omega$.

The function $\frac{1}{v}$ is also positive and continuous on $\partial\Omega$, as before, there exists a function G in $H^\infty_\delta(\Omega)$ with $|G|^2 = \frac{1}{v}$ on $\partial\Omega$. Since $\log \frac{1}{v} = -\log v$, it easy to verify that index of G is exactly $(\gamma_1^{-1}, \gamma_2^{-1}, ..., \gamma_n^{-1})$ and hence δ is equal to γ^{-1}. Evidently $FG = 1$ on Ω. □

Now we establish the bijective correspondence which preserve the unitary equivalence class, promised earlier. From (3.1), we know that the harmonic measure $d\omega_p$ is of the form $h^2 ds$ for some positive continuous function h on $\partial\Omega$. Combining this with the preceding Lemma, we see that there is a F in $H^\infty_\gamma(\Omega)$ with $|F|^2 = h^2$ on $\partial\Omega$ and a G in $H^\infty_{\gamma^{-1}}(\Omega)$ with $|G|^2 = h^{-2}$ on $\partial\Omega$. Now consider the map $M_F : \big(H^2_\alpha(\Omega), d\omega_p(z)\big) \mapsto \big(H^2_{\alpha\gamma}(\Omega), ds\big)$, defined by the equation

$$M_F(g) = Fg, \quad g \in \big(H^2_\alpha(\Omega), d\mu_\zeta(z)\big).$$

Clearly, M_F is a unitary operator and its inverse is the operator M_G. The multiplication operator M_F intertwines the corresponding operator of multiplication by the coordinate function on the Hilbert spaces $\big(H^2_\alpha(\Omega), d\omega_p(z)\big)$ and $\big(H^2_{\alpha\gamma}(\Omega), ds\big)$ establishing a bijective correspondence of the unitary equivalence classes of bundle shifts. As a consequence we have the following theorem which was proved by Abrahamse and Douglas (see [2, Theorem 5 and 6]) with the harmonic measure $d\omega_p$ instead of the arc length measure ds.

Theorem 3.2. *The bundle shift T_α on $\big(H^2_\alpha(\Omega), ds\big)$ is unitarily equivalent to the bundle shift T_β on $\big(H^2_\beta(\Omega), ds\big)$ if and only if $\alpha = \beta$.*

With the help of techniques used in proving Lemma 3.1, now we provide an alternative proof of the following characterization theorem for pure, rationally cyclic, subnormal operator with spectrum equal to $\overline{\Omega}$ and whose normal spectrum lies in $\partial\Omega$, proved by Abrahamse and Douglas (see [2, Theorem 11]). Let $d\omega_p$ be the harmonic measure on $\partial\Omega$ w.r.t a fixed but arbitrary point p in Ω.

Theorem 3.3 (Abrahamse and Douglas)**.** *Every pure, rationally cyclic subnormal operator with spectrum equal to $\overline{\Omega}$ and whose normal spectrum lies in $\partial\Omega$, is unitarily equivalent to a bundle shift T_α on $\big(H^2_\alpha(\Omega), d\omega_p\big)$ for some character α.*

Proof. Let T be a rationally cyclic, subnormal operator with spectrum equal to $\overline{\Omega}$ and normal spectrum of T lies in the boundary of Ω. It is well known that each such T is unitarily equivalent to the multiplication operator M on $\big(H^2(\partial\Omega), d\mu\big)$ for some measure μ, namely the scalar spectral measure of the minimal normal extension N of T, supported on $\partial\Omega$ (see [11, p.51, Theorem 5.2].) Here by $\big(H^2(\partial\Omega), d\mu\big)$, we mean closure of $\mathrm{Rat}(\overline{\Omega})$ in $\big(L^2(\partial\Omega), d\mu\big)$.

Since T is a rationally cyclic pure subnormal operator, the scalar spectral measure of the minimal normal extension N is mutually absolutely continuous w.r.t harmonic measure $d\omega_p$ (see [2, Proposition 3.3].) So we have $d\mu = h d\omega_p$, for some non negative measurable function h in $\big(L^1(\partial\Omega), d\omega_p\big)$. It is also known that $\log h \in \big(L^1(\partial\Omega), d\omega_p\big)$. Otherwise, $\big(H^2(\partial\Omega), d\mu\big)$ would be equal to $\big(L^2(\partial\Omega), d\mu\big)$ (see [2, lemma 3.1]) and in that case T would be a normal operator.

Since the boundary of Ω consists of analytic jordan curve, dirichlet problem is solvable. So solving dirichlet problem with boundary value $\frac{1}{2}\log h$, we will have a harmonic function u on Ω so that boundary value of u equal to $\frac{1}{2}\log h$ a.e. For this u, we have a multiple valued conjugate harmonic function v on Ω. Note that v has a period

$$p_i = -\int_{\partial\Omega_i} \frac{\partial u}{\partial \eta} ds, \quad i = 1, 2, \ldots, n.$$

over the negatively oriented boundary component $\partial\Omega_i$. Now consider the multiplicative function F on Ω defined by $F := \exp(u + iv)$. Clearly F is a bounded, invertible multiplicative function whose index is given by the following n tuple of number of unit modulus

$$\big(\exp(ip_1), \exp(ip_2), \ldots, \exp(ip_n)\big) = \alpha \ \text{(say)}.$$

Note that we also have, $|F|^2 = \exp(2u) = h$ a.e. on $\partial\Omega$. Consequently we have $d\mu = |F|^2 d\omega_p$, for some multiplicative function F of index α. Now consider linear map $U : \big(H^2(\partial\Omega), d\mu\big) \mapsto \big(H^2_\alpha(\Omega), d\omega_p\big)$ defined by $U(f) = Ff$. Clearly, U is an unitary map. The operator U being a multiplication operator, it intertwines the corresponding multiplication operator. This gives us that T is unitarily equivalent to the bundle shift T_α on $\big(H^2_\alpha(\Omega), d\omega_p\big)$. $\square$

It can be shown using the result of Abrahamse and Douglas (see [2, Theorem 3]) that for any character α, the adjoint of the rank 1 bundle shift T_α lies in $B_1(\Omega^*)$. Since the bundle shifts T_α is subnormal, it follows that the adjoint of the bundle shifts T_α admits Ω^* as a spectral set. Consequently, we have an inequality for the curvature of the bundle shifts, namely,

$$\mathscr{K}_{T_\alpha^*}(w) \leq -4\pi^2 (S_{\Omega^*}(w,w))^2,\ w \in \Omega^*.$$

Given any fixed but arbitrary point ζ in Ω, in the following section, we recall the proof (slightly different from the original proof given in [26] of the existence of a bundle shift T_α for which equality occurs at $\bar{\zeta}$ in the curvature inequality. However, the main theorem of this section is the "uniqueness" of such an operator.

Theorem 3.4 (Uniqueness)**.** *If the bundle shift T_α on $\left(H^2_\alpha(\Omega), ds\right)$ and the bundle shift T_β on $\left(H^2_\beta(\Omega), ds\right)$ are extremal at the point $\bar{\zeta}$, that is, if they satisfy*

$$\mathscr{K}_{T_\alpha^*}(\bar{\zeta}) = -4\pi^2 (S_{\Omega^*}(\bar{\zeta},\bar{\zeta}))^2 = \mathscr{K}_{T_\beta^*}(\bar{\zeta}),$$

then the bundle shifts T_α and T_β are unitarily equivalent, which is the same as $\alpha = \beta$.

The Hardy space $\left(H^2(\Omega), d\omega_p\right)$ consists of holomorphic function on Ω such that $|f|^2$ has a harmonic majorant on Ω. Each f in $\left(H^2(\Omega), d\omega_p\right)$ has a non tangential boundary value almost everywhere. In the usual way $\left(H^2(\Omega), d\omega_p\right)$ is identified with a closed subspace of $L^2(\partial\Omega, d\omega_p)$ (see [33, Theorem 3.2]). Let λ be a positive continuous function on $\partial\Omega$. As the measure λds and the harmonic measure $d\omega_p$ on $\partial\Omega$ are boundedly mutually absolutely continuous one can define an equivalent norm on $\left(H^2(\Omega)\right.$ in the following way

$$\|f\|^2_{\lambda ds} = \int_{\partial\Omega} |f(z)|^2 \lambda(z) ds(z).$$

Let $\left(H^2(\Omega), \lambda ds\right)$ denote the linear space $H^2(\Omega)$ endowed with the norm λds. Since the harmonic measure $d\omega_p$ is boundedly mutually absolutely continuous w.r.t the arc length measure ds and λ is a positive continuous function on $\partial\Omega$, it follows that the $\|\cdot\|_{\lambda ds}$ defines an equivalent norm on $\left(H^2(\Omega), d\omega_p\right)$. So, the identity map $id : \left(H^2(\Omega), d\omega_p\right) \mapsto \left(H^2(\Omega), \lambda ds\right)$ is an invertible map intertwining the associated multiplication operator M. Thus $\left(H^2(\Omega), \lambda ds\right)$ acquires the structure of a Hilbert space and the operator M on it is rationally cyclic, pure subnormal, its spectrum is equal to $\overline{\Omega}$ and finally its normal spectrum is equal to $\partial\Omega$. Consequently, the operator M on $\left(H^2(\Omega), \lambda ds\right)$ must be unitarily equivalent to the bundle shift T_α on $\left(H^2_\alpha(\Omega), ds\right)$ for some character α. Now, we compute the character α.

Since λ is a positive continuous function on $\partial\Omega$, using Lemma 3.1, we have the existence of a character α and a function F in $H^\infty_\alpha(\Omega)$ satisfying $|F|^2 = \lambda$ on $\partial\Omega$. The function F is also invertible in the sense that there exist a function G in $H^\infty_{\alpha^{-1}}(\Omega)$ such that $FG = 1$ on Ω. It is straightforward to verify that the linear map $M_F : \big(H^2(\Omega), \lambda ds\big) \mapsto \big(H^2_\alpha(\Omega), ds\big)$ defined by

$$M_F(g) = Fg, \quad g \in \big(H^2(\Omega), \lambda ds\big)$$

is unitary. Also M_F being a multiplication operator, intertwines the corresponding multiplication operator by the coordinate function on the respective Hilbert spaces. From Lemma 3.1, it is clear that the character α is determined by the following n tuple of numbers:

$$c_j(\lambda) = -\int_{\partial\Omega_j} \frac{\partial}{\partial\eta_z}\big(u_\lambda(z)\big)ds(z), \ \ \text{for } j = 1,2,...,n, \tag{3.2}$$

where u_λ is the harmonic function on Ω with continuous boundary value $\frac{1}{2}\log\lambda$. Using this information along with the Theorem 3.2, we deduce the following Lemma which describe the unitary equivalence class of the multiplication operator M on $\big(H^2(\Omega), \lambda ds\big)$.

Lemma 3.5. *Let λ, μ be two positive continuous function on $\partial\Omega$. Then the operators M on the Hilbert spaces $\big(H^2(\Omega), \lambda ds\big)$ and $\big(H^2(\Omega), \mu ds\big)$ are unitarily equivalent if and only if*

$$\exp\big(ic_j(\lambda)\big) = \exp\big(ic_j(\mu)\big), \quad j = 1,\ldots,n.$$

Remark 3.6. The case when $\lambda(z)$ is a positive constant on each of the boundary component $\partial\Omega_i$ is of special interest (cf. [10]). Let us assume

$$\lambda(z) = \begin{cases} \exp(2\lambda_j),\ z \in \partial\Omega_j,\ j = 1,2,\ldots,n, \\ 1,\ z \in \partial\Omega_{n+1} \end{cases} \tag{3.3}$$

where the constants λ_j are real numbers. In this case, $u_\lambda(z)$, the harmonic extension of $\frac{1}{2}\log\lambda(z)$ equals to $\lambda_1\omega_1(z) + \cdots + \lambda_n\omega_n(z)$, where $\omega_j(z)$ is the harmonic function on Ω whose boundary value on $\partial\Omega_j$ is 1 and it is 0 on every other boundary component. Recall that the period matrix $(\!(p_{i,j})\!)$ associated to the domain Ω, where is given by the formula

$$p_{i,j} = -\int_{\partial\Omega_i} \frac{\partial}{\partial\eta_z}\omega_j(z)ds.$$

It is well known that the period matrix $P = (\!(p_{i,j})\!)$ is positive definite (cf. [30, page 39]). Now if we denote the column vector $(\lambda_1,\ldots,\lambda_n)^t$ by $\hat{\lambda}$, then it is easy to see that $c_i(\lambda) = \sum_{j=1}^n p_{i,j}\lambda_j = (P\hat{\lambda})_i$.

So, for weight functions λ (and μ) of the form (3.3), an equivalent form of the Lemma 3.5 merits a special mention:

Let λ and μ be two positive continuous function on the boundary of Ω, of the above type, then the operators M on the Hilbert spaces $\big(H^2(\Omega), \lambda ds\big)$ and $\big(H^2(\Omega), \mu ds\big)$ are unitarily equivalent if and only if $P(\hat{\lambda} - \hat{\mu}) \in \mathbb{Z}^n$.

It also follows from a result of Abrahamse (see [1, Proposition 1.15]) that given a character α there exist a invertible element F in $H^\infty_\alpha(\Omega)$ such that

$$|F(z)|^2 = \begin{cases} 1, & \text{if } z \in \partial\Omega_{n+1} \\ p_j, & \text{if } z \in \partial\Omega_j, \ \ j = 1, \cdots, n, \end{cases}$$

where p_j are positive constant. Thus we have proved the following theorem.

Theorem 3.7. *Given any character α, there exists a positive continuous function λ defined on $\partial\Omega$ such that the operator M on $\left(H^2(\Omega), \lambda ds\right)$ is unitarily equivalent to the bundle shift T_α on $\left(H^2_\alpha(\Omega), ds\right)$.*

3.2 Weighted Kernel and Extremal Operator at a fixed point

Let λ be a positive continuous function on $\partial\Omega$. Since $\left(H^2(\Omega), d\omega_p\right)$ is a reproducing kernel Hilbert space and the norm on $\left(H^2(\Omega), d\omega_p\right)$ is equivalent to the norm on $\left(H^2(\Omega), \lambda ds\right)$, it follows that $\left(H^2(\Omega), \lambda ds\right)$ is also a reproducing kernel Hilbert space. Let $K^{(\lambda)}(z, w)$ denote the kernel function for $\left(H^2(\Omega), \lambda ds\right)$.

The case $\lambda \equiv 1$ gives us the Sżego kernel $S(z, w)$ for the domain Ω. Associated to the Sżego kernel, there exists a conjugate kernel $L(z, w)$, called the Garabedian kernel, which is related to the Sżeego kernel via the following identity.

$$\overline{S(z, w)}ds = \frac{1}{i}L(z, w)dz, \ \ w \in \Omega \text{ and } \ z \in \partial\Omega.$$

We recall several well known properties of these two kernels when $\partial\Omega$ consists of jordan analytic curves. For each fixed w in Ω, the function $S_w(z)$ is holomorphic in a neighbourhood of Ω and $L_w(z)$ is holomorphic in a neighbourhood of $\Omega - \{w\}$ with a simple pole at w. $L_w(z)$ is non vanishing on $\overline{\Omega} - \{w\}$. The function $S_w(z)$ is non vanishing on $\partial\Omega$ and has exactly n zero in Ω (cf. [7, Theorem 13.1]). In [29, Theorem 1] Nehari has extended these result for the kernel $K^{(\lambda)}(z, w)$.

Theorem 3.8 (Nehari)**.** *Let Ω be a bounded domain in the complex plane, whose boundary consists of $n+1$ analytic jordan curve and let λ be a positive continuous function on $\partial\Omega$. Then there exist two analytic function $K^{(\lambda)}(z, w)$ and $L^{(\lambda)}(z, w)$ with the following properties: for each fixed w in Ω, the function $K^{(\lambda)}_w(z)$ and $L^{(\lambda)}_w(z) - (2\pi(z-w))^{-1}$ are holomorphic in Ω; $|K^{(\lambda)}_w(z)|$ is continuous on $\overline{\Omega}$ and $|L^{(\lambda)}_w(z)|$ is continuous in $\overline{\Omega} - C_\epsilon$, where C_ϵ denotes a small open disc about w; $K^{(\lambda)}_w(z)$ and $L^{(\lambda)}_w(z)$ are connected by the identity*

$$\overline{K^{(\lambda)}_w(z)}\lambda(z)ds = \frac{1}{i}L^{(\lambda)}_w(z)dz, \ \ w \in \Omega \text{ and } z \in \partial\Omega. \tag{3.4}$$

These properties determine both functions uniquely.

From (3.4), we have that $\frac{1}{i}K_w^{\lambda}(z)L_w^{\lambda}(z)dz \geq 0$. The boundary $\partial\Omega$ consists of Jordan analytic curves, therefore from the Schwartz reflection principle, it follows that the function K_w^{λ} and $L_w^{\lambda} - (2\pi(z-w))^{-1}$ are holomorphic in a neighbourhood of $\overline{\Omega}$.

We have shown that the operator M on $\big(H^2(\Omega), \lambda ds\big)$ is unitarily equivalent to a bundle shift of rank 1. Consequently the adjoint operator M^* lies in $B_1(\Omega^*)$ admitting $\overline{\Omega^*}$ as a spectral set from which a curvature inequality follows:

$$\mathcal{K}_T(w) \leq -4\pi^2(S_{\Omega^*}(w,w))^2,\ w \in \Omega^*.$$

Or equivalently,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w) \geq 4\pi^2(S_{\Omega}(w,w))^2, \quad w \in \Omega.$$

Fix a point ζ in Ω. The following lemma provides a criterion for the adjoint operator M^* on $\big(H^2(\Omega), \lambda ds\big)$ to be extremal at $\bar{\zeta}$, that is,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w)\,|_{w=\zeta} = 4\pi^2(S_{\Omega}(\zeta,\zeta))^2.$$

Lemma 3.9. *The operator M^* on the Hilbert space $\big(H^2(\Omega), \lambda ds\big)$ is extremal at $\bar{\zeta}$ if and only if $L_{\zeta}^{(\lambda)}(z)$ and the Szëgo kernel at ζ, namely $S_{\zeta}(z)$ have the same set of zeros in Ω.*

Proof. Consider the closed convex set M_1 in $\big(H^2(\Omega), \lambda(z)ds\big)$ defined by

$$M_1 := \{f \in \big(H^2(\Omega), \lambda(z)ds\big) : f(\zeta) = 0, f'(\zeta) = 1\}.$$

Now consider the extremal problem is

$$\inf\{\|f\|^2 : f \in M_1\}. \tag{3.5}$$

Since M_1 is a closed convex set, there exist a unique function F in M_1 which solve the extremal problem. Let H_1 be the closed subspace of $\big(H^2(\Omega), \lambda ds\big)$ defined by

$$H_1 := \{f \in \big(H^2(\Omega), \lambda(z)ds\big) : f(\zeta) = 0, f'(\zeta) = 0\} = \big(\mathrm{Span}\{\mathrm{K}_{\zeta}^{(\lambda)}, \bar{\partial}\mathrm{K}_{\zeta}^{(\lambda)}\}\big)^{\perp}.$$

Since $f + g \in M_1$, whenever $f \in M_1$ and $g \in H_1$, It is evident that the unique function F which solves the extremal problem must belong to $H_1^{\perp}$. Let $F = c_1K_{\zeta}^{(\lambda)} + c_2\bar{\partial}K_{\zeta}^{(\lambda)}$ be the solution of the extremal problem. Since $F \in M_1$, we have

$$c_1K^{(\lambda)}(\zeta,\zeta) + c_2\bar{\partial}K^{(\lambda)}(\zeta,\zeta) = 0,$$
$$c_1\partial K^{(\lambda)}(\zeta,\zeta) + c_2\bar{\partial}\partial K^{(\lambda)}(\zeta,\zeta) = 1.$$

Let G denotes the Grammian matrix for the vectors $\{K_\zeta^{(\lambda)}, \bar{\partial} K_\zeta^{(\lambda)}\}$, that is,

$$G = \begin{pmatrix} K^{(\lambda)}(\zeta,\zeta) & \bar{\partial} K^{(\lambda)}(\zeta,\zeta) \\ \partial K^{(\lambda)}(\zeta,\zeta) & \bar{\partial}\partial K^{(\lambda)}(\zeta,\zeta) \end{pmatrix}$$

and c denotes the column vector $(c_1, c_2)^{tr}$. we have $Gc = (0,1)^{tr} = e_2$. Thus $c = G^{-1}e_2$. Consequently, we have that,

$$\begin{aligned} \|F\|^2 &= \|c_1 K_\zeta^{(\lambda)} + c_2 \bar{\partial} K_\zeta^{(\lambda)}\|^2 \\ &= \langle Gc, c\rangle \\ &= \langle G^{-1}e_2, e_2\rangle \\ &= \Big\{K^{(\lambda)}(\zeta,\zeta)\Big(\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w)|_{w=\zeta}\Big)\Big\}^{-1}. \end{aligned}$$

Now consider the function g in $\big(H^2(\Omega), \lambda(z)ds\big)$ defined by

$$g(z) := \frac{K_\zeta^{(\lambda)}(z) F_\zeta(z)}{2\pi S(\zeta,\zeta) K^{(\lambda)}(\zeta,\zeta)}, \quad z \in \Omega,$$

where $F_\zeta(z) = \frac{S_\zeta(z)}{L_\zeta(z)}$ denote the Ahlfors map for the domain Ω at the point ζ (see [7, Theorem 13.1]). Using the reproducing property for the kernel function $K^{(\lambda)}$ and the fact that $|F_\zeta(z)| \equiv 1$ on $\partial\Omega$, it is straightforward to verify that

$$\|g\|^2_{\lambda ds} = \Big(K^{(\lambda)}(\zeta,\zeta) 4\pi^2 S(\zeta,\zeta)^2\Big)^{-1}.$$

Since $F_\zeta(\zeta) = 0$ and $F'_\zeta(\zeta) = 2\pi S(\zeta,\zeta)$, it follows that $g \in M_1$. Consequently we have,

$$\Big(K^{(\lambda)}(\zeta,\zeta) 4\pi^2 S(\zeta,\zeta)^2\Big)^{-1} \geq \Big\{K^{(\lambda)}(\zeta,\zeta)\Big(\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w)|_{w=\zeta}\Big)\Big\}^{-1}.$$

Or equivalently,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w)\,|_{w=\zeta} \geq 4\pi^2 (S_\Omega(\zeta,\zeta))^2.$$

So equality holds if and only if g solve the extremal problem in (3.5) if and only if g is orthogonal to the subspace H_1. Hence, we conclude that the operator M^* on the Hilbert space $\big(H^2(\Omega), \lambda(z)ds\big)$ is extremal at $\bar{\zeta}$ if and only if g is orthogonal to the subspace H_1. Now consider the following integral

$$\begin{aligned}
I_f &= \int_{\partial\Omega} f(z)\overline{K_\zeta^{(\lambda)}(z)}\,\overline{F_\zeta(z)}\lambda(z)ds \\
&= \frac{1}{i}\int_{\partial\Omega} f(z)\overline{F_\zeta(z)}L_\zeta^{(\lambda)}(z)dz \qquad \text{(Using the identity 3.4)} \\
&= \frac{2\pi}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{F_\zeta(z)}L_\zeta^{(\lambda)}(z)dz \\
&= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)L_\zeta^{(\lambda)}(z)\big(2\pi L_\zeta(z)\big)}{S_\zeta(z)}dz.
\end{aligned}$$

Since $H_1 \cap \mathrm{Rat}(\overline{\Omega})$ is dense in H_1, g is orthogonal to H_1 if and only if I_f vanishes for all $f \in H_1 \cap \mathrm{Rat}(\overline{\Omega})$. Observe that we have $L_\zeta^{(\lambda)}(z)L_\zeta(z)$ is holomorphic in $\Omega - \{\zeta\}$ with a pole of order 2 at ζ. As $\partial\Omega$ consists of Jordan analytic curve, both the function $L_(^{(\lambda)}\zeta)(z)$ and $L_\zeta(z)$ are also holomorphic in a neighbourhood of $\partial\Omega$. It is known that $L_\zeta(z)$ has no zero in $\overline{\Omega} - \{\zeta\}$ and $S_\zeta(z)$ has exactly n zero say $a_1, a_2, ..., a_n$ in Ω, (cf. [7,]).

Now we claim that I_f vanishes for all $f \in H_1 \cap \mathrm{Rat}(\overline{\Omega})$ if and only if the set of zeros of the function $L_\zeta^{(\lambda)}(z)$ in Ω is $\{a_1, a_2, \ldots, a_n\}$.

First if we assume that $L_\zeta^{(\lambda)}(z)$ has $\{a_1, a_2, \ldots, a_n\}$ as the zero set in Ω, then the integrand in I_f is holomorphic in a neighbourhood of $\overline{\Omega}$ for every f in $H_1 \cap \mathrm{Rat}(\overline{\Omega})$ and consequently I_f vanishes for every f in $H_1 \cap \mathrm{Rat}(\overline{\Omega})$. Conversely if $L_\zeta^{(\lambda)}(z)$ doesn't vanish at one of a_j's, without loss of generality, say at a_1, then the function

$$f(z) = (z-\zeta)^2 \prod_{k=2}^{n} (z - a_k)$$

is in $H_1 \cap \mathrm{Rat}(\overline{\Omega})$. Observe that the integrand in I_f, with this choice of the function f, is holomorphic in a neighbourhood of $\overline{\Omega}$ except at the point a_1, where it has a simple pole. So by the Residue theorem the integral I_f equals the residue of the integrand at a_1, which is not zero completing the proof. □

3.2.1 Existence of Extremal operator

We provide below two different descriptions of an extremal operator at $\bar{\zeta}$ using the criterion obtained in Lemma 3.9 Let $a_1, a_2, ..., a_n$ be the zeros of the Szëgo $S_\zeta(z)$ in Ω.

Realization of the extremal operator at $\bar{\zeta}$

Consider the function λ on $\partial\Omega$ defined by

$$\lambda(z) := \prod_{k=1}^{n} |z - a_k|^2, \ z \in \partial\Omega.$$

Then, for $z \in \partial\Omega$, we have

$$\frac{\overline{S_\zeta(z)}}{\prod_{j=1}^{n}(\bar{z}-\bar{a}_j)(\zeta-a_j)}\lambda(z)ds = \frac{\prod_{k=1}^{n}(z-a_k)}{\prod_{k=1}^{n}(\zeta-a_k)}\overline{S_\zeta(z)}ds$$
$$= \frac{1}{i}\frac{\prod_{k=1}^{n}(z-a_k)}{\prod_{k=1}^{n}(\zeta-a_k)}L_\zeta(z)dz$$

Note that the function $S_\zeta(z)\Big(\prod_{j=1}^{n}(z-a_j)(\bar{\zeta}-\bar{a}_j)\Big)^{-1}$ is holomorphic in a neighborhood of $\overline{\Omega}$ and the function $L_\zeta(z)\Big(\prod_{k=1}^{n}(z-a_k)\Big)\Big(\prod_{k=1}^{n}(\zeta-a_k)\Big)^{-1}$ is a meromorphic in a neighbourhood of $\overline{\Omega}$ with a simple pole at ζ. Hence using the uniqueness portion of the Theorem (3.8), we get

$$K_\zeta^{(\lambda)}(z) = \frac{S_\zeta(z)}{\prod_{j=1}^{n}(z-a_j)(\bar{\zeta}-\bar{a}_j)},\ z\in\overline{\Omega} \text{ and } L_\zeta^{(\lambda)}(z) = \frac{\prod_{k=1}^{n}(z-a_k)}{\prod_{k=1}^{n}(\zeta-a_k)}L_\zeta(z),\ \ z\in\overline{\Omega}-\{\zeta\}.$$

Clearly, $\{a_1, a_2, ..., a_n\}$ is the zero set of the function $L_\zeta^{(\lambda)}(z)$. So, the adjoint operator M^* on $\big(H^2(\Omega), \lambda(z)ds\big)$ is an extremal operator at $\bar{\zeta}$.

A second realization of the extremal operator at $\bar{\zeta}$:

This realization of the extremal operator was obtained earlier in [26]. Consider the measure

$$\lambda(z)ds = \frac{|S_\zeta(z)|^2}{S(\zeta,\zeta)}ds,\ z\in\partial\Omega,$$

on the boundary $\partial\Omega$. Using the reproducing property of the Sżego kernel, it is easy to verify that

$$\langle f, 1\rangle_{\left(H^2(\Omega),\lambda ds\right)} = f(\zeta).$$

This gives us $K_\zeta^{(\lambda)}(z) = 1$ for all $z \in \overline{\Omega}$. So we have

$$\lambda(z)ds = \frac{S_\zeta(z)}{S(\zeta,\zeta)}\overline{S_\zeta(z)}ds,\ \ z\in\partial\Omega$$
$$= \frac{1}{i}\frac{S_\zeta(z)}{S(\zeta,\zeta)}L_\zeta(z)dz,\ \ \ z\in\partial\Omega.$$

Now the function $S_\zeta(z)L_\zeta(z)\big(S(\zeta,\zeta)\big)^{-1}$ is a meromorphic function in a neighbourhood of $\overline{\Omega}$ with a simple pole at ζ. Again, using the uniqueness guaranteed in Theorem (3.8), we get

$$L_\zeta^{(\lambda)}(z) = S_\zeta(z)L_\zeta(z)\big(S(\zeta,\zeta)\big)^{-1}, \quad z \in \overline{\Omega} - \{\zeta\}.$$

Again, the zero set of the function $L_\zeta^{(\lambda)}(z)$ is equal to the set $\{a_1, a_2, ..., a_n\}$. So the operator M^* on $\big(H^2(\Omega), \lambda(z)ds\big)$ is an extremal operator at $\bar{\zeta}$.

We shall prove that the any extremal operator which is also the adjoint of a bundle shift is uniquely determined up to unitary equivalence. An amusing consequence of this uniqueness is that the two realizations of the extremal operators given above must coincide up to unitary equivalence.

3.2.2 Index of the Blaschke product

To facilitate the proof of the uniqueness, we need to recall basic properties of multiplicative Blaschke product on Ω and its index of automorphy. This is also going to be a crucial ingredient in determining the character α of the extremal operator at $\bar{\zeta}$.

Let $g(z,a)$ be the Green's function for the domain Ω, whose critical point is $a \in \Omega$. The multiplicative Blaschke factor with zero at a, is defined as follows

$$B_a(z) = \exp(-g(z,a) - ig^*(z,a)), \text{ for all } z \in \Omega,$$

where $g^*(z,a)$ is the multivalued conjugate of the Green's function $g(z,a)$, which is harmonic on $\Omega - \{a\}$. So, $B_a(z)$ is a multiplicative function on Ω, which vanishes only at the point a with multiplicity 1 and on $\partial\Omega$ its absolute value is identically 1. Note that periods of the conjugate harmonic function $g^*(z,a)$ around the boundary component $\partial\Omega_j$ is equal to

$$p_j(a) = -\int_{\partial\Omega_j} \frac{\partial}{\partial\eta_z}\big(g(z,a)\big)ds_z, \text{ for } j = 1,2,...,n.$$

The negative sign appearing in the equation for the periods is a result of the assumption that $\partial\Omega$ is positively oriented, that is, the boundary $\partial\Omega_j$, $j = 1,2,\ldots,n$, except the outer one are oriented in clockwise direction.

Since the Blaschke factor $B_a(z)$ is multiplicative function on Ω, therefore it is induced by a modulus automorphic function on unit disc, say b_α, for some α. The character α uniquely determines n-tuple of complex number of unit modulus. These are the image under α of the generators of the group G of Deck transformations relative to the covering map $\pi : \mathbb{D} \to \Omega$. This n - tuple, called the index of the Blaschke factor $B_a(z)$, is of the form

$$\{\exp(-ip_1(a)), \exp(-ip_2(a), \cdots, \exp(-ip_n(a))\}.$$

We recall bellow the well known relationship of the period $p_j(a)$ to the harmonic measure $\omega_j(z)$ of the boundary component $\partial\Omega_j$, namely,

$$\omega_j(a) = -\frac{1}{2\pi}\int_{\partial\Omega_j}\frac{\partial}{\partial\eta_z}\big(g(z,a)\big)ds_z = \frac{1}{2\pi}p_j(a), \text{ for } j = 1,2,...,n,$$

where the harmonic measure $\omega_j(z)$ is the function which is harmonic in Ω and has the boundary values 1 on $\partial\Omega_j$ and is 0 on all the other boundary components. Hence the index of the Blaschke factor $B_a(z)$ is

$$\operatorname{ind}(B_a(z)) = \{\exp(-2\pi i\omega_1(a)), \exp(-2\pi i\omega_2(a), \cdots, \exp(-2\pi i\omega_n(a))\}.$$

For each of these n tuple of numbers, there exist a homomorphism $\alpha : G \to \mathbb{T}$ such that these n tuple of numbers occur as the image of the n generator of the group G under the map α completing the bijective correspondence between the character α and the index. It follows that the function $B_a := b_\alpha \circ \pi^{-1}$ lies in H^∞_α.

The index of the Blaschke product $B(z) = \prod_{k=1}^{m} B_{a_k}(z)$, $a_k \in \Omega$, is equal to

$$\operatorname{ind}(B(z)) = \Big\{\exp\big(-2\pi i\sum_{k=1}^{m}\omega_1(a_k)\big), \cdots, \exp\big(-2\pi i\sum_{k=1}^{m}\omega_n(a_k)\big)\Big\}. \tag{3.6}$$

3.2.3 Zeros of the Szëgo kernel

Fixing ζ in Ω, which is $n+1$ - connected, as pointed out earlier, the Szëgo kernel $S_\zeta(z)$ has exactly n zeros (counting multiplicity) in Ω. Let $a_1, a_2, \ldots, a_n$ be the zeros of $S_\zeta(z)$. Hence the Ahlfors function $F_\zeta(z)$ at the point ζ has exactly $n+1$ zeros in Ω, namely $\zeta, a_1, a_2, \ldots, a_n$. Now an interesting relation between the points $a_1, \ldots, a_n$ and ζ becomes evident.

First consider the Blaschke product $B(z) = B_\zeta(z).\prod_{k=1}^{n} B_{a_k}(z)$. The index of the Blaschke product $B(z)$, using (3.6), is easily seen to be of the form

$$\beta = (\beta_1, \beta_2, \cdots, \beta_n), \text{ where } \beta_j = \Big\{\exp\Big(-2\pi i\big(\omega_j(\zeta) + \sum_{k=1}^{n}\omega_j(a_k)\big)\Big)\Big\} \text{ for } j = 1,2,\ldots,n.$$

The Ahlfors function $F_\zeta(z)$ is in $H^\infty(\Omega)$ and it is holomorphic in a neighbourhood of $\bar\Omega$ as long as the boundary $\partial\Omega$ is analytic. Therefore in the inner outer factorization of $F_\zeta(z)$, there is no singular inner function and it follows that

$$|F_\zeta(z)| = |B(z)||\psi(z)|, \ z \in \Omega,$$

where $\psi(z)$ is a multiplicative outer function of index

$$\beta^{-1} = (\beta_1^{-1}, \beta_2^{-1}, \cdots, \beta_n^{-1}).$$

Now consider the linear map $L : \big(H^2(\Omega), ds(z)\big) \mapsto \big(H^2_{\beta^{-1}}(\Omega), ds\big)$, defined by

$$Lf = \psi f, \quad f \in \big(H^2(\Omega), ds(z)\big).$$

Note that $\psi(z)$ is outer and it is bounded in absolute value (since $F_\zeta(z)$ is bounded) on Ω. It is straightforward to verify that L is a unitary operator. Also, since L is a multiplication operator, it intertwines any two multiplication operators on the respective Hilbert spaces.

As a corollary of Theorem 3.2, we must have $\beta^{-1} = (1, 1, \cdots, 1)$. This implies

$$\exp\Big(-2\pi i\big(\omega_j(\zeta) + \sum_{k=1}^{n} \omega_j(a_k)\big)\Big) = 1, \quad j = 1, 2, \ldots, n, \tag{3.7}$$

relating the point ζ to the zeros $a_1, a_2, \cdots, a_n$ of the Sžego kernel $S_\zeta(z)$.

3.2.4 Uniqueness of the Extremal operator

Assume that for a positive continuous function λ on $\partial\Omega$, the operator M^* on the Hilbert space $\big(H^2(\Omega), \lambda(z)ds\big)$ is extremal at $\bar{\zeta}$. The function $K_\zeta^{(\lambda)}(z)$ is analytic in a neighborhood of $\overline{\Omega}$ and the conjugate kernel $L_\zeta^{(\lambda)}(z)$ is meromorphic in a neighborhood of $\overline{\Omega}$ with a simple pole only at the point ζ. Also from Lemma 3.9, we have that the zero set of $L_\zeta^{(\lambda)}(z)$ is the the set $\{a_1, a_2, ..., a_n\}$, where $\{a_1, a_2, ..., a_n\}$ are the zeros of $S_\zeta(z)$ in Ω. We have, using the equation (3.4), that

$$|K_\zeta^{(\lambda)}(z)|^2 \lambda(z) ds = \frac{1}{i} K_\zeta^{(\lambda)}(z) L_\zeta^{(\lambda)}(z) dz, \quad z \in \partial\Omega.$$

An application of the Generalized Argument Principle shows that the total number of zeros of $K_\zeta^{(\lambda)}(z)$ and $L_\zeta^{(\lambda)}(z)$ in $\overline{\Omega}$, where a zero on the boundary is counted as $\frac{1}{2}$, is equal to n. Hence it follows that $a_1, a_2, ..., a_n$ are the all zeros of $L_\zeta^{(\lambda)}(z)$ in $\overline{\Omega}$ and $K_\zeta^{(\lambda)}(z)$ has no zero in $\overline{\Omega}$.

Nehari [29, Theorem 4] has shown that the meromorphic function

$$R(z) = \frac{K_\zeta^{(\lambda)}(z)}{L_\zeta^{(\lambda)}(z)}, \quad z \in \overline{\Omega}$$

with exactly one zero at ζ and poles exactly at $a_1, a_2, ..., a_n$, solves the extremal problem

$$\sup\{|f'(\zeta)| : f \in B_\lambda\},$$

where B_λ denotes the class of meromorphic function on Ω. Each f in B_λ is required to vanish at ζ and it is assumed that the set of poles of f is a subset of $\{a_1, a_2, ..., a_n\}$. The radial limit of the functions f at $z_0 \in \partial\Omega$, from within Ω, in the class B_λ are uniformly bounded:

$$\limsup_{z \to z_0} |f(z)| \le \frac{1}{\lambda(z_0)}, \quad z_0 \in \partial\Omega.$$

The proof includes the verification

$$|R(z)| = \frac{1}{\lambda(z)}, \; z \in \partial\Omega.$$

Now consider the multiplicative function G on Ω defined by

$$G(z) = \frac{B_\zeta(z)}{R(z)\prod_{j=1}^{n} B_{a_j}(z)}, \quad z \in \overline{\Omega}.$$

So, G is a multiplicative function in a neighbourhood of $\overline{\Omega}$. Also by construction $|G|$ has no zero in $\overline{\Omega}$. Using the inner outer factorization for multiplicative functions (cf. [41, Theorem 1]), we see that G is a bounded multiplicative outer function. Also note that

$$|G(z)| = \lambda(z), \quad z \in \partial\Omega.$$

The index of G is given by

$$\Big\{\exp\Big(2\pi i\big(-\omega_1(\zeta) + \sum_{j=1}^{n}\omega_1(a_j)\big)\Big), \ldots, \exp\Big(2\pi i\big(-\omega_n(\zeta) + \sum_{j=1}^{n}\omega_n(a_j)\big)\Big)\Big\}.$$

Using equation (3.7), we infer that the index of $G(z)$ must be equal to

$$\Big\{\exp\Big(-4\pi i\omega_1(\zeta)\Big), \ldots, \exp\Big(-4\pi i\omega_n(\zeta)\Big)\Big\}.$$

The function G is outer and hence the function $F := \sqrt{G}$ is well defined. It is a bounded multiplicative outer function with $|F(z)|^2 = \lambda(z)$ for all z in $\partial\Omega$. Let's denote the index of F by

$$\Big\{\exp\Big(-2\pi i\omega_1(\zeta)\Big), \ldots, \exp\Big(-2\pi i\omega_n(\zeta)\Big)\Big\}.$$

Now consider the linear map $V : \big(H^2(\Omega), \lambda(z)\,ds\big) \mapsto \big(H^2_\alpha(\Omega), ds\big)$ defined by

$$Vf = Ff, \quad f \in \big(H^2(\Omega), \lambda(z)\,ds\big).$$

It is easily verified that V is a unitary multiplication operator, which intertwines the corresponding multiplication operators on the respective Hilbert spaces. Hence the character α for the bundle shift T_α on $\big(H^2_\alpha(\Omega), ds\big)$, which is extremal at $\bar{\zeta}$, is uniquely determined by the following n tuple of complex number of unit modulus:

$$\begin{aligned}&\big\{\exp\big(-2\pi i\omega_1(\zeta)\big), \ldots, \exp\big(-2\pi i\omega_n(\zeta)\big)\big\}\\ &\qquad\qquad = \big\{\exp\big(2\pi i(1-\omega_1(\zeta))\big), \ldots, \exp\big(2\pi i(1-\omega_n(\zeta))\big)\big\}.\end{aligned}$$

Hence if the adjoint of a bundle shift (upto unitary equivalence) is extremal at $\bar{\zeta}$, then it is uniquely determined. This completes the proof of the Theorem 3.4.

Since the group of the Deck transformations G for the covering $\pi : \mathbb{D} \to \Omega$ is isomorphic to the free group on n generators, any character α of the group G is unambiguously determined, up to a permutation in the choice of generators for the group G, by the n-tuple $\{x := (x_1, x_2, \cdots, x_n) : x_1, \ldots, x_n \in [0,1)\}$, namely,

$$\alpha(g_k) = \exp(2\pi i x_k),\ x_k \in [0,1)\ 1 \le k \le n,$$

where g_k, $1 \le k \le n$, are generators of the group G. The unitary equivalence class of the bundle shifts T_α of rank 1 is therefore determined by the n-tuple x in $[0,1)^n$ corresponding to the character α.

The character corresponding to the n-tuple $\Big((1-\omega_1(\zeta)), (1-\omega_2(\zeta)), \cdots, (1-\omega_n(\zeta))\Big)$ defines the bundle shift which is extremal at $\bar{\zeta}$. Let $\phi : \Omega \to [0,1)^n$ be the induced map, that is,

$$\phi(\zeta) = \Big((1-\omega_1(\zeta)), (1-\omega_2(\zeta)), \cdots, (1-\omega_n(\zeta))\Big).$$

Suita [36] shows that the map ϕ is not onto since $(0, \ldots, 0)$, which corresponds to the operator M^* on the usual Hardy space, cannot be in its range. However, we show below that many other bundle shifts are missing from the range of the map ϕ, when $n \ge 2$.

Let $\omega_{n+1}(z)$ be the harmonic measure for the outer boundary component $\partial\Omega_{n+1}$. Thus ω_{n+1} is the harmonic function on Ω which is 1 on $\partial\Omega_{n+1}$ and is 0 on all the other boundary components. We have

$$\sum_{j=1}^{n+1} \omega_j \equiv 1 \text{ and } 0 < \omega_{n+1}(z) < 1,\ z \in \Omega,$$

therefore

$$(n-1) < \sum_{j=1}^{n} \big(1 - \omega_j(\zeta)\big) < n.$$

From this, for $n \ge 2$, it follows that the set of extremal operators does not include the adjoint of many of the bundle shifts. For instance, if the index of a bundle shift is $(x_1, \ldots, x_n)$ in $[0,1)^n$ is such that $x_1 + \cdots + x_n < n-1$, then it cannot be an extremal operator at any $\bar{\zeta}$, $\zeta \in \Omega$.

3.3 The special case of the Annulus

Let Ω be an Annulur domain $A(0; R, 1)$ with inner radius R, $0 < R < 1$, and outer radius 1. In this case we have a explicit expression for the harmonic measure corresponding to the boundary component $\partial\Omega_1$, namely,

$$\omega_1(z) = \frac{\log|z|}{\log R}.$$

So for a fixed point ζ in $A(0;R,1)$, the character of the unique bundle shift which happens to be an extremal operator at $\bar{\zeta}$ is determined by the number

$$\alpha(\zeta) = \exp\Big(2\pi i\big(1-\omega_1(\zeta)\big)\Big).$$

From this expression for the index, it is clear, in the case of an Annulur domain $A(0;R,1)$, that the adjoint of every bundle shift except the trivial one, is an extremal operator at some point $\bar{\zeta}$ in Ω^*. In fact this is true of any doubly connected bounded domain Ω with Jordan analytic boundary since for such domain we have $\omega_1(\Omega) = (0,1)$, where ω_1 is the harmonic measure corresponding to the inner boundary component $\partial\Omega_1$.

We now give a different proof of the Theorem 3.4 in the case of $\Omega = A(0;R,1)$. In the course of this proof we see the effect of the weights on the zeros of the weighted Hardy kernels $K^{(\alpha)}$. This question was raised in [24].

For a fixed real number α, Consider the measure $\mu_\alpha ds$ on the boundary of the Annulus, where the function μ_α is defined by

$$\mu_\alpha(z) = \begin{cases} 1, & \text{if } |z| = 1, \\ R^{2\alpha}, & |z| = R. \end{cases}$$

It is straightforward to verify that the function $\{f_n(z)\}_{n\in\mathbb{Z}}$ defined by

$$f_n(z) = \frac{z^n}{\sqrt{2\pi(1+R^{2\alpha+2n+1})}}, \quad n \in \mathbb{Z},$$

forms an orthonormal basis for the Hilbert space $\big(H^2(\Omega), \mu_\alpha ds\big)$. The function

$$K^{(\alpha)}(z,w) := \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{(z\bar{w})^n}{1+R^{2\alpha+2n+1}}, \quad z, w \in \Omega,$$

is uniformly convergent on compact subsets of Ω. Hence $K^{(\alpha)}$ is the reproducing kernel of the Hilbert space $\big(H^2(\Omega), \mu_\alpha ds\big)$. For each fixed w in Ω, the kernel function $K^{(\alpha)}(z,w)$ is defined on Ω. However, it extends analytically to a larger domain. To describe this extension, recall that the Jordan Kronecker function, introduced by Venkatachaliengar (cf. [39, p.37]), is given by the formula

$$f(b,t) = \sum_{k\in\mathbb{Z}} \frac{t^n}{1-bR^{2n}}.$$

This series converges for $R^2 < |t| < 1$, and for all $b \neq R^{2k}, k \in \mathbb{Z}$. Venkatachaliengar, using Ramanujan's ${}_1\psi_1$ summation formula, has established the following identity (cf. [39, p. 40])

$$f(b,t) = \frac{\prod_{j=0}^{\infty}(1-btR^{2j})\prod_{j=0}^{\infty}(1-\frac{R^{2j+2}}{bt})\prod_{j=0}^{\infty}(1-R^{2j+2})\prod_{j=0}^{\infty}(1-R^{2j+2})}{\prod_{j=0}^{\infty}(1-tR^{2j})\prod_{j=0}^{\infty}(1-\frac{R^{2j+2}}{t})\prod_{j=0}^{\infty}(1-bR^{2j})\prod_{j=0}^{\infty}(1-\frac{R^{2j+2}}{b})}. \tag{3.8}$$

This extends the definition of $f(b,t)$, as a meromorphic function, to all of the complex plane with a simple poles at $b = R^{2k}$, $t = R^{2k}$; $k \in \mathbb{Z}$. For an arbitrary but fixed point w in Ω, since the function $f(-R^{2\alpha+1}, z\bar{w})$ coincides with $2\pi K^{(\alpha)}(z,w)$ for all z in Ω, and f is a meromorphic on the entire complex plane, it follows that $K_w^{(\alpha)}$ also extends to all of $\mathbb{C}$ as a meromorphic function. The poles of $K_w^{(\alpha)}$ are exactly at $\frac{R^{2k}}{\bar{w}}, k \in \mathbb{Z}$. The zeros of the kernel function $K_w^{(\alpha)}(z)$ in Ω can also be computed using the equation 3.8. The zeros (b,t) of the function f must satisfy one of the following identities

$$bt = R^{-2j}, j = 0,1,2,\ldots \quad \text{or,} \quad bt = R^{2j+2}, j = 0,1,2,\ldots$$

For example, when $\alpha = 0$, the kernel $K^{(\alpha)}(z,w)$ is the Sżego kernel $S(z,w)$. It follows that if w is a fixed but arbitrary point in Ω, then the zero set of the Sżego kernel function $S_w(z)$ is $\{-\frac{R}{\bar{w}}\}$.

The operator M on the Hilbert space $\left(H^2(\Omega), \mu_\alpha ds\right)$ is a bilateral weighted shift with weight sequence

$$\omega_n^{(\alpha)} = \sqrt{\frac{1+R^{2\alpha+2n+3}}{1+R^{2\alpha+2n+1}}}, \quad n \in \mathbb{Z}.$$

The identity

$$\omega_n^{(\alpha+1)} = \omega_{n+1}^{(\alpha)}, \ \ n \in \mathbb{Z},$$

makes the operators M on $\left(H^2(\Omega), \mu_\alpha ds\right)$ and $\left(H^2(\Omega), \mu_{\alpha+1} ds\right)$ unitarily equivalent. Thus there is a natural map from the unitary equivalence classes of these bi-lateral shifts onto $[0,1)$. In the case of the annulus $A(0;R,1)$, we find that u_{μ_α} and $c_1(\mu_\alpha)$, as defined in equation (3.2), are equal to $\alpha \log|z|$ and $2\pi\alpha$ respectively. Applying Lemma 3.5, we see that the operators M on $\left(H^2(\Omega), \mu_\alpha ds\right)$ and $\left(H^2(\Omega), \mu_{\alpha+1} ds\right)$ are unitarily equivalent if and only if $\alpha - \beta$ is an integer. Thus we have a bijective correspondence between the unitary equivalence classes of these bi-lateral shifts and $[0,1)$, and we may assume without loss of generality that $\alpha \in [0,1)$.

For each $\alpha \in [0,1)$, the operator M on $\left(H^2(\Omega), \mu_\alpha ds\right)$ is unitarily equivalent to the bundle shift T_β on $\left(H^2_\beta(\Omega), ds\right)$, where the character β is determined by the unimodular scalar $\exp(2\pi i \alpha)$.

Now Fix a point ζ in Ω. It is known that $S_\zeta(z)$, the Sžego kernel at ζ for the domain Ω has exactly one zero at $-\frac{R}{\bar\zeta}$. The existence of a conjugate kernel $L^{(\alpha)}(z,w)$ is established in [29]. Then using the characterization for the extremal operator at $\bar\zeta$, it follows that the operator M^* on $\big(H^2(\Omega),\mu_\alpha ds\big)$ is extremal at $\bar\zeta$ if and only if $L^{(\alpha)}_\zeta(-\frac{R}{\bar\zeta})=0$. From the identity

$$zL^{(\alpha)}(z,w)=K^{(\alpha)}(\tfrac{1}{z},\bar w)$$

proved in [24, p.1118], and recalling that $K^{(\alpha)}_{\bar\zeta}(-\frac{\bar\zeta}{R})=\sum\limits_{k\in\mathbb{Z}}\frac{(-\frac{|\zeta|^2}{R})^n}{1+R^{2\alpha+2n+1}}$, we conclude: The operator M^* is extremal at $\bar\zeta$ if and only if

$$\sum_{k\in\mathbb{Z}}\frac{(-\frac{|\zeta|^2}{R})^n}{1+R^{2\alpha+2n+1}}=0.$$

Consequently, the operator M^* on $\big(H^2(\Omega),\mu_\alpha ds\big)$ is extremal at $\bar\zeta$ if and only if the Jordan Kronecker function f satisfy

$$f(-R^{2\alpha+1},-\tfrac{|\zeta|^2}{R})=0.$$

So for a fixed ζ, the real number $\alpha\in[0,1)$ must satisfy at least one of these identities

$$R^{2\alpha}|\zeta|^2=R^{-2j},\ j=0,1,2,...;\text{or } R^{2\alpha}|\zeta|^2=R^{2j+2},\ j=0,1,2,\ldots$$

In any case, one must have

$$\alpha=\Big(1-\frac{\log|\zeta|}{\log R}\Big)\ (\mathrm{mod}\ 1).$$

So the unitary equivalence class of an operator which is extremal at $\bar\zeta$, and is the adjoint of a bundle shift is uniquely determined. Hence we have proved the Theorem stated below.

Theorem 3.10. *The operator M^* on $\big(H^2(\Omega),\mu_\alpha ds\big)$ is an extremal operator at the point $\bar\zeta$ in Ω^* if and only if $\alpha=\big(1-\frac{\log|\zeta|}{\log R}\big)$ (mod 1).*

Chapter 4

Generalized Curvature Inequality

4.1 Curvature and Local operators

Let Ω be a bounded domain in $\mathbb{C}^m$. Set $\Omega^* = \{\bar{z} : z \in \Omega\}$. Cowen and Douglas also introduced the class $B_n(\Omega^*)$, for a bounded domain $\Omega^* \subseteq \mathbb{C}^m$ (cf. [15]). Curto and Salinas have studied this class with more detail (cf. [16]). We recall the definition here.

Definition 4.1. A m-tuple of commuting bounded operators $\boldsymbol{T} = (T_1, T_2, \ldots, T_m)$ on a complex separable Hilbert space $\mathscr{H}$ is said to be in $B_n(\Omega^*)$ if

1. for $w = (w_1, w_2, \ldots, w_m)$ in Ω^*, the dimension of the joint kernel $\bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k)$ is equal to n.
2. for $w = (w_1, w_2, \ldots, w_m)$ in Ω^*, the operator $\mathscr{D}_{\boldsymbol{T}-w} : \mathscr{H} \mapsto \mathscr{H} \oplus \mathscr{H} \oplus \ldots \oplus \mathscr{H}$ defined by $\mathscr{D}_{\boldsymbol{T}-w}(h) = \oplus_{k=1}^m (T_k - w_k)h$, must have closed range.
3. $\bigvee_{w \in \Omega^*} \left(\bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k)\right) = \mathscr{H}$

Like one variable case, these above conditions also ensure the existence of a rank n holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$ over Ω^*, that is,

$$E_T := \{(w, v) \in \Omega^* \times \mathscr{H} : v \in \big(\bigcap_{k=1}^m \operatorname{Ker}(T_k - w_k)\big)\}, \pi(w, v) = w.$$

The equivalence class of holomorphic Hermitian bundle E_T and the joint unitary equivalence class of the tuple of operators $\boldsymbol{T}$ determine each other.

Theorem 4.2 (Cowen-Douglas)**.** *Two tuple of operators $\boldsymbol{T}$ and $\boldsymbol{S}$ in $B_n(\Omega^*)$ are jointly unitarily equivalent if and only if the associated holomorphic Hermitian vector bundles $E_{\boldsymbol{T}}$ and $E_{\boldsymbol{S}}$ are locally equivalent.*

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\boldsymbol{T} = (T_1, T_2, \dots, T_m)$ be a commuting tuple of bounded operators on some separable complex Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ lies in $B_n(\Omega^*)$. Let $E_{\boldsymbol{T}}$ be the associated holomorphic Hermitian vector bundle over Ω^* and $\mathsf{K} = \mathsf{K}(E_{\boldsymbol{T}}, D)$ be the curvature associated with canonical connection D of the holomorphic Hermitian vector bundle $E_{\boldsymbol{T}}$. Let $\mathscr{K} := \left(\!\left(\mathscr{K}_{i,j}\right)\!\right)_{i,j=1}^m$, where $\mathscr{K}_{i,j}$ is a C^∞ cross section of $\mathrm{Hom}(E_{\boldsymbol{T}}, E_{\boldsymbol{T}})$ such that

$$\mathsf{K}(\sigma) = \sum_{i,j=1}^m \mathscr{K}_{i,j}(\sigma)\, dz_i \wedge d\bar{z}_j,$$

for all C^∞ smooth section σ of $E_{\boldsymbol{T}}$. Let $w = (w_1, \dots, w_m)$ be an arbitrary but fixed point in Ω^*. Let $\gamma(z) = (\gamma_1(z), \dots, \gamma_n(z))$ be a local holomorphic frame of $E_{\boldsymbol{T}}$ in a neighbourhood of w say $\Omega_0^* \subset \Omega^*$. Then the matrix of the metric of the bundle $E_{\boldsymbol{T}}$ at $z \in \Omega_0^*$ w.r.t the frame γ is given by

$$h_\gamma(z) = \left(\!\left(\langle \gamma_j(z), \gamma_i(z)\rangle\right)\!\right)_{i,j=1}^n.$$

We write $\partial_i = \frac{\partial}{\partial z_i}$ and $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}$. Then $\mathscr{K}_{i,j}$ w.r.t the frame γ is of the form

$$\mathscr{K}_{i,j}(\gamma)(z) = -\bar{\partial}_j\Big((h_\gamma(z))^{-1}\big(\partial_i h_\gamma(z)\big)\Big),\ z \in \Omega_0^*.$$

Since

$$(T_j - w_j)\gamma_i(w) = 0, \quad i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m, \tag{4.1}$$

it follows that,

$$(T_j - w_j)(\partial_k \gamma_i(w)) = \gamma_i(w)\delta_{j,k}, \quad i = 1, 2, \dots, n, \text{ and } j, k = 1, \dots, m, \tag{4.2}$$

For any natural number $k = (p-1)n + q$, $1 \le p \le m+1$, and $1 \le q \le n$, set $v_k := \partial_{p-1}(\gamma_q(w))$. Also, it will be useful to let $\mathbf{v}_i := \big(v_{(i-1)n+1}, \dots, v_{(i-1)n+n}\big)$. Thus $\mathbf{v}_i$ is also $\partial_{i-1}\gamma$, where $\gamma = (\gamma_1, \dots, \gamma_n)$. We have that v_k, $1 \le k \le (m+1)n$, forms a basis for the subspace

$$\mathscr{M}_w = \bigcap_{i,j=1}^m \ker(T_i - w_i)(T_j - w_j).$$

Let N_w be the tuple of nilpotent operator $(N_1, \dots, N_m)$ defined by $N_i(w) = (T_i - w_i)\mid_{\mathscr{M}_w}$. We denote the block operator matrix $\left(\!\left(N_i(w)N_j(w)^*\right)\!\right)_{i,j=1}^m$ by $N_w N_w^*$.

Proposition 4.3. *There exists an orthonormal basis in the subspace $\mathscr{M}_w$ such that the matrix representation of $N_w N_w^*$ with respect to this basis takes the form*

$$\begin{pmatrix} -(\mathscr{K}_{\tilde{\gamma}}(w))^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{\gamma}$ is a frame in a neighbourhood of w which is also orthonormal at the point w.

Proof. Let P be an invertible matrix of size $(m+1)n \times (m+1)n$. Let

$$(\mathbf{u}_1, \dots \mathbf{u}_{m+1}) = (\mathbf{v}_1, \dots, \mathbf{v}_{m+1}) \begin{pmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,m+1} \\ P_{2,1} & P_{2,2} & \dots & P_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m+1,1} & P_{m+1,2} & \dots & P_{m+1,m+1} \end{pmatrix},$$

where each $P_{i,j}$ is a $n \times n$ matrix. Clearly, $(\mathbf{u}_1, \dots, \mathbf{u}_{m+1})$ is a basis, not necessarily orthonormal, in the subspace $\mathscr{M}_w$.

The set of vectors $\{\mathbf{u} = (\mathbf{u}_1, \dots \mathbf{u}_{m+1})\}$ is an orthonormal basis in $\mathscr{M}_w$ if and only if the scalar matrix P satisfy $P\bar{P}^t = G^{-1}$, where G denotes the $(m+1)n \times (m+1)n$, grammian matrix $(\!(\langle v_j, v_i \rangle)\!)$.

$$G = \begin{pmatrix} h & \partial_1 h & \dots & \partial_m h \\ \bar{\partial}_1 h & \bar{\partial}_1 \partial_1 h & \dots & \bar{\partial}_1 \partial_m h \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\partial}_m h & \bar{\partial}_m \partial_1 h & \dots & \bar{\partial}_m \partial_m h \end{pmatrix}.$$

One choice of P is an upper triangular matrix which actually comes from Gram-Schmidt orthogonalization process. There are other choices like $G^{-1/2}$. Let us fix an invertible matrix P which is (block) upper triangular and assume that $P\bar{P}^t = G^{-1}$. Following equation (4.1) and (4.2), the matrix representation of N_l w.r.t. the basis $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_{m+1})$ is

$$\left[N_l\right]_{\mathbf{v}} = \begin{pmatrix} 0_{n\times n} & \dots & 0_{n\times n} & I_{n\times n}((1, l+1)\text{th block}) & \dots & 0_{n\times n} \\ 0_{n\times n} & \dots & 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & \dots & 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \end{pmatrix},$$

for $l = 1, 2, \dots, m$. Therefore w.r.t the orthonormal basis $(\mathbf{u}_1, \dots, \mathbf{u}_{m+1})$, the matrix of N_l will be of the form

$$\begin{aligned} \left[N_l\right]_{\mathbf{u}} &= P^{-1} \left[N_l\right]_{\mathbf{v}} P \\ &= \begin{pmatrix} 0_{n\times n} & A_l^1 & \dots & A_l^m \\ 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \end{pmatrix}, \end{aligned}$$

where each A_l^i is a square matrix of size n, for $l, i = 1, 2, \dots, m$. It is now evident that for $l, r = 1, 2, \dots, m$, we have

$$\left[N_l N_r^*\right]_{\mathbf{u}} = Q \left[N_l\right]_{\mathbf{v}} G^{-1} \left[N_r\right]_{\mathbf{v}} \overline{Q}^t,$$

where $Q = P^{-1}$. To continue, we write the matrix G^{-1} in the form of a block matrix:

$$G^{-1} = \begin{pmatrix} \star_{n\times n} & \star_{n\times mn} \\ \star_{mn\times n} & R_{mn\times mn} \end{pmatrix} \tag{4.3}$$
$$= \begin{pmatrix} \star_{n\times n} & \star_{n\times n} & \star_{n\times n} & \dots & \star_{n\times n} \\ \star_{n\times n} & R_{1,1} & R_{1,2} & \dots & R_{1,m} \\ \star_{n\times n} & R_{2,1} & R_{2,2} & \dots & R_{2,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \star_{n\times n} & R_{m,1} & R_{m,2} & \dots & R_{m,m} \end{pmatrix}_{m+1,m+1},$$

where each $R_{i,j}$ is a $n \times n$ matrix. Then we have

$$\left[N_l N_r^*\right]_{\mathbf{u}} = \begin{pmatrix} Q_{1,1} R_{l,a} \overline{Q}_{1,1}^t & 0_{n\times mn} \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}.$$

Since P is (block) upper triangular, we have $\mathbf{u}_1 = \mathbf{v}_1 P_{1,1}$, that is,

$$(u_1, u_2, \dots, u_n) \quad = \quad (v_1, v_2, \dots, v_n) P_{1,1}.$$

The vectors in the tuple $\mathbf{u}_1$ are orthonormal and hence $P_{1,1}$ must satisfy $P_{1,1}\overline{P}_{1,1}^t = h^{-1}$, or equivalently, $\overline{Q}_{1,1}^t Q_{1,1} = h$. Using the polar decomposition, we have the decomposition $Q_{1,1} = U_Q\sqrt{h}$ for some unitary matrix U_Q. Using the unitary U_Q, define a new orthonormal basis, say, $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_{m+1})$ for $\mathscr{M}_w$:

$$\begin{aligned}(\mathbf{y}_1, \dots, \mathbf{y}_{m+1}) &= (\mathbf{u}_1, \dots, \mathbf{u}_{m+1}) \begin{pmatrix} U_Q & 0_{n\times mn} \\ 0_{mn\times n} & I_{mn\times mn} \end{pmatrix} \\ &= (\gamma(w), \mathbf{v}_2, \dots, \mathbf{v}_n) \begin{pmatrix} \sqrt{h^{-1}} & \star_{n\times mn} \\ 0_{mn\times n} & \star_{mn\times mn} \end{pmatrix} \\ &= (\tilde{\gamma}(w), \mathbf{u}_2, \dots, \mathbf{u}_{m+1}).\end{aligned} \tag{4.4}$$

where $\tilde{\gamma}(w) := \gamma(w)\sqrt{h^{-1}}$. Hence w.r.t the orthonormal basis $(\mathbf{y}_1, \dots, \mathbf{y}_{m+1})$ of the subspace $\mathscr{M}_w$, the linear transformation $N_l N_r^*$ has the matrix representation

$$[N_l N_r^*]_{\mathbf{y}} = \begin{pmatrix} \sqrt{h} R_{l,a} \sqrt{h} & 0_{n\times mn} \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}. \tag{4.5}$$

Note that matrix representation of N_l w.r.t. the orthonormal basis $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_{m+1})$ for the subspace $\mathscr{M}_w$, is of the form

$$\left[N_l\right]_{\mathbf{y}} = \begin{pmatrix} 0_{n\times n} & \mathbf{t}_l^1(w) & \dots & \mathbf{t}_l^m(w) \\ 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \end{pmatrix} = \begin{pmatrix} 0_{n\times n} & \mathbf{t}_l(w) \\ 0_{mn\times n} & 0_{mn\times mn} \end{pmatrix}, \tag{4.6}$$

where each $\mathbf{t}_l^i(w)$ is a square matrix of size n, for $l, i = 1, 2, \ldots, m$. Let $\boldsymbol{t}(w)$ be the $mn \times mn$ matrix given by

$$\boldsymbol{t}(w) = \begin{pmatrix} \mathbf{t}_1(w) \\ \mathbf{t}_2(w) \\ \vdots \\ \mathbf{t}_m(w) \end{pmatrix}.$$

Now following equation (4.5), we then have

$$\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}} = (\sqrt{h} \otimes I_m) R (\sqrt{h} \otimes I_m). \tag{4.7}$$

From equation (4.5), we find the block matrix representation for $N_w N_w^*$. So we have

$$\begin{aligned} [N_w N_w^*]_{\mathbf{y}} &= \left(\left[N_l N_r^* \right]_{\mathbf{y}} \right)_{l,r=1}^m \\ &= \left(\begin{pmatrix} \sqrt{h} R_{l,r} \sqrt{h} & 0_{n \times mn} \\ 0_{mn \times n} & 0_{mn \times mn} \end{pmatrix} \right)_{l,r=1}^m \\ &\simeq \begin{pmatrix} \left(\!\left(\sqrt{h} R_{l,a} \sqrt{h} \right)\!\right)_{l,a=1}^m & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{4.8}$$

where $\simeq$ denotes the unitary equivalence. Note that this unitary equivalence will be obtained by changing order of the orthonormal basis from $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{m+1})$ to $\tilde{\mathbf{y}} = (\mathbf{y}_1, \tilde{\mathbf{y}}_2, \ldots, \tilde{\mathbf{y}}_{m+1})$ in an appropriate manner. Thus

$$[N_w N_w^*]_{\tilde{\mathbf{y}}} = \begin{pmatrix} (\sqrt{h} \otimes I_m) R (\sqrt{h} \otimes I_m) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

Also note that in this change order of the first n vector in the orthonormal basis remains unchanged.

The Grammian G admits a natural decomposition as a 2×2 block matrix, namely,

$$G = \begin{pmatrix} h & \partial_1 h & \ldots & \partial_m h \\ \bar{\partial}_1 h & \bar{\partial}_1 \partial_1 h & \ldots & \bar{\partial}_1 \partial_m h \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\partial}_m h & \bar{\partial}_m \partial_1 h & \ldots & \bar{\partial}_m \partial_m h \end{pmatrix} = \begin{pmatrix} h_{n \times n} & X_{n \times mn} \\ L_{mn \times n} & S_{mn \times mn} \end{pmatrix}.$$

Computing the 2×2 entry of the inverse of this block matrix and equating it to R, we have

$$\begin{aligned} R^{-1} &= S - L h^{-1} X \\ &= \left(\!\left(\bar{\partial}_i \partial_j h \right)\!\right)_{i,j=1}^m - \left(\!\left((\bar{\partial}_i h) h^{-1} (\partial_j h) \right)\!\right)_{i,j=1}^m \\ &= \left(\!\left(h \bar{\partial}_i (h^{-1} \partial_j h) \right)\!\right) \\ &= -\left(\!\left(h \mathscr{K}_{i,j}(\gamma)(w) \right)\!\right) \\ &= -(h \otimes I_m) \mathscr{K}_\gamma(w), \end{aligned}$$

where $\mathcal{K}_{i,j}(\gamma)(w)$ denote the matrix of the curvature $\mathcal{K}_{i,j}$ at $w\in\Omega_0^*$ w.r.t the frame γ of the bundle $E_{\boldsymbol{T}}$ on Ω_0^* and $\mathcal{K}_\gamma(w)=\big(\!\big(\mathcal{K}_{i,j}(\gamma)(w)\big)\!\big)_{i,j=1}^m$. Hence we have

$$R \;=\; \big(-\mathcal{K}_\gamma(w)\big)^{-1}(h^{-1}\otimes I_m). \tag{4.10}$$

Combining equation (4.9) and (4.10), we conclude that

$$\begin{aligned}[N_wN_w^*]_{\tilde{\mathbf{y}}} &= \begin{pmatrix}(\sqrt{h}\otimes I_m)\big(-\mathcal{K}_\gamma(w)\big)^{-1}(\sqrt{h}\otimes I_m)^{-1} & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}\big(-\mathcal{K}_{\tilde\gamma}(w)\big)^{-1} & 0\\ 0 & 0\end{pmatrix},\end{aligned} \tag{4.11}$$

where $\tilde\gamma(z)$ is the new holomorphic frame on Ω_0^* defined by

$$\tilde\gamma(z)=\gamma(z)\sqrt{h_\gamma(w)^{-1}}=\gamma(z)\sqrt{h^{-1}},\;\; z\in\Omega_0^*.$$

and using the transformation rule for the curvature with respect to two frame on Ω_0^*, namely, $\mathcal{K}_{i,j}(\tilde\gamma)(z)=\sqrt{h}(\mathcal{K}_{i,j}(\gamma)(z))\sqrt{h^{-1}}$, for all $z\in\Omega_0^*$, we get that

$$\mathcal{K}_{\tilde\gamma}(w)=(\sqrt{h}\otimes I_m)(\mathcal{K}_\gamma(w))(\sqrt{h}\otimes I_m)^{-1}.$$

Since $h_{\tilde\gamma}(w)=I$, the frame $\tilde\gamma$ is orthonormal at w. □

Note that following equation (4.7) and (4.10), we also have

$$\boldsymbol{t}(w)\overline{\boldsymbol{t}(w)}^{\mathrm{tr}}=-(\mathcal{K}_{\tilde\gamma}(w))^{-1}. \tag{4.12}$$

As an application, it is easy to obtain a curvature inequality for those commuting tuples of operators $\boldsymbol{T}$ in the Cowen-Douglas class $B_n(\Omega^*)$ which admit $\overline{\Omega}^*$ as a spectral set. This is easily done via the holomorphic functional calculus. Since the matrix representation of N_l w.r.t. the orthonormal basis $\mathbf{y}=(\mathbf{y}_1,\ldots,\mathbf{y}_{m+1})$ for the subspace $\mathcal{M}_w$, is of the form

$$\Big[N_l\Big]_{\mathbf{u}}=\begin{pmatrix}0_{n\times n} & \mathbf{t}_l\\ 0_{mn\times n} & 0_{mn\times mn}\end{pmatrix}.$$

Now it is easy to see that for any holomorphic function f defined on some open neighbourhood of the compact set $\overline{\Omega}^*$, we have

$$\begin{aligned}f(\boldsymbol{T})_{|\mathcal{M}_w} &= f\big(\boldsymbol{T}_{|\mathcal{M}_w}\big)\\ &= \begin{pmatrix}f(w) & \nabla f(w)\cdot\boldsymbol{t}(w)\\ 0 & f(w)\end{pmatrix}=f(\boldsymbol{T}_w),\end{aligned}$$

where $\boldsymbol{T}_w$ is the m tuple of operator $\boldsymbol{T}_{|\mathscr{M}_w}$ and

$$\begin{aligned}\nabla f(w)\cdot \boldsymbol{t}(w) &= \partial_1 f(w)\mathbf{t}_1(w)+\cdots+\partial_m f(w)\mathbf{t}_m(w)\\ &= \big((\partial_1 f(w))I_n,\ldots,(\partial_m f(w))I_n\big)(\boldsymbol{t}(w))\\ &= (I_n\otimes \nabla f(w))(\boldsymbol{t}(w)).\end{aligned}$$

If $\boldsymbol{T}$ admits $\overline{\Omega}^*$ as a spectral set, then the inequality $I - f(\boldsymbol{T}_w)^* f(\boldsymbol{T}_w)\geq 0$ is evident for all holomorphic functions mapping $\overline{\Omega}^*$ to the unit disc $\mathbb{D}$. As is well-known, we may assume without loss of generality that $f(w)=0$. Consequently, the inequality $I - f(\boldsymbol{T}_w)^* f(\boldsymbol{T}_w)\geq 0$ with $f(w)=0$ is equivalent to

$$\Big(I_n\otimes \overline{\nabla f(w)}^{\mathrm{tr}}\Big)(I_n\otimes \nabla f(w)) \leq -(\mathscr{K}_{\tilde{\gamma}}(w)). \tag{4.13}$$

Let $V\in\mathbb{C}^{mn}$ be a vector of the form

$$V=\begin{pmatrix}V_1\\ \vdots\\ V_m\end{pmatrix},\ \text{where}\ V_i=\begin{pmatrix}V_i(1)\\ \vdots\\ V_i(n)\end{pmatrix}\in\mathbb{C}^n.$$

The carathéodory norm of the tangent vector $V\in\mathbb{C}^{mn}$ is defined by

$$\begin{aligned}(C_{\Omega,w}(V))^2 &= \sup\big\{\langle\Big(I_n\otimes \overline{\nabla f(w)}^{\mathrm{tr}}\Big)(I_n\otimes\nabla f(w))V,V\rangle : f\in\mathscr{O}(\overline{\Omega}), \|f\|_\infty\leq 1, f(w)=0\big\}\\ &= \sup\big\{\sum_{i,j=1}^m \overline{\partial_i f(w)}\partial_j f(w)\langle V_j,V_i\rangle : f\in\mathscr{O}(\overline{\Omega}), \|f\|_\infty\leq 1, f(w)=0\big\}.\end{aligned}$$

Thus from equation (4.13), a proof of the theorem below follows.

Theorem 4.4. *Let $\boldsymbol{T}$ be a commuting tuple of operator in $B_n(\Omega)$ admitting $\overline{\Omega}^*$ as a spectral set. Then for an arbitrary but fixed point $w\in\overline{\Omega}^*$, there exist a frame $\tilde{\gamma}$ of the bundle $E_{\boldsymbol{T}}$, defined in a neighbourhood of w, which is orthonormal at w, so that following inequality holds*

$$\langle\mathscr{K}_{\tilde{\gamma}}(w)V,V\rangle \leq -(C_{\Omega,w}(V))^2,\ \textit{for every}\ \ V\in\mathbb{C}^{mn}.$$

4.2 Curvature inequality for operators of higher rank

In this section, using the Proposition 4.3, we derive a curvature inequality specializing to the case of a bounded planar domains Ω^*. The computations in this case are direct and somewhat more transparent, therefore for the sake of clarity, we repeat below what has been already said in the general case.

Let T be an operator in $B_n(\Omega^*)$ and assume that it admits $\overline{\Omega}^*$ as a spectral set. One obtains the familiar curvature inequality for such an operator T by noting that the restriction of the operator T to the invariant subspace $\mathcal{M}_w = \ker(T-w)^2$, $w \in \Omega^*$, must also admit $\overline{\Omega}^*$ as a spectral set. For an operator T in $B_n(\Omega^*)$, the restriction to the invariant subspace $\mathcal{M}_w = \ker(T-w)^2$, $w \in \Omega^*$, admits $\overline{\Omega}^*$ as a spectral set as before. This gives rise to the generalized curvature inequality. Uchiyama (cf. [38]) was the first one to prove a curvature inequality for operators in $B_n(\mathbb{D})$ using techniques from Sz.-Nagy – Foias model theory for contractions. He then obtained a generalization to domains $\Omega \subseteq \mathbb{D}$. However, the inequality we obtain below follows directly from the functional calculus applied to the local operators. More recently, K. Wang and G. Zhang (cf. [42]) have obtained a series of very interesting (higher order) curvature inequalities for operators in $B_1(\Omega)$.

Let T be an operator in $B_n(\Omega^*)$ with $\overline{\Omega}^*$ as a spectral set for T. Fix a point $w \in \Omega^*$. As in the previous section, we have $\gamma(z) = (\gamma_1(z), \ldots, \gamma_n(z))$ is a local holomorphic frame of E_T, in a neighbourhood of w say $\Omega_0^* \subset \Omega^*$ and the metric of the bundle E_T at $z \in \Omega_0^*$ w.r.t the frame γ is given by $h_\gamma(z) = \big(\!\big(\langle \gamma_j(z), \gamma_i(z)\rangle\big)\!\big)_{i,j=1}^n$, for all $z \in \Omega_0^*$. We have that the matrix representation of $N_w = (T-w)|_{\mathcal{M}_w}$ w.r.t the ordered basis

$$(\gamma(w), \gamma'(w)) = (\mathbf{v}_1, \mathbf{v}_2) = (\gamma_1(w), \ldots, \gamma_n(w), \gamma_1'(w), \ldots, \gamma_n'(w))$$

is of the form

$$[N_w]_{\mathbf{v}} = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix}.$$

The Grammian G corresponding to this basis is of the form

$$G \;=\; \begin{pmatrix} h & \partial h \\ \bar{\partial} h & \bar{\partial}\partial h \end{pmatrix},$$

where the matrix h is given by $h = h_\gamma(w)$. Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ be an orthonormal basis of $\ker(T-w)^2$ obtained from any invertible block upper triangular matrix P satisfying $P\bar{P}^t = G^{-1}$ as follows:

$$(\mathbf{u}_1, \mathbf{u}_2) := (\mathbf{v}_1, \mathbf{v}_2)P = (\mathbf{v}_1, \mathbf{v}_2)\begin{pmatrix} P_{1,1} & P_{1,2} \\ 0 & P_{2,2} \end{pmatrix}.$$

The matrix of N_w w.r.t this basis is of the form

$$\begin{aligned} [N_w]_{\mathbf{u}} &= P^{-1}[N_w]_{\gamma(w),\gamma'(w)}P \\ &= \begin{pmatrix} 0_{n\times n} & A \\ 0_{n\times n} & 0_{n\times n} \end{pmatrix}. \end{aligned}$$

where the matrix A is given by $A = P_{1,1}^{-1}P_{2,2}$. From the equation (4.4), we have that

$$(\mathbf{y}_1, \mathbf{y}_2) \;=\; (\mathbf{u}_1, \mathbf{u}_2)\begin{pmatrix} U_Q & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{pmatrix}$$

is also an orthonormal basis for $\mathscr{M}_w$ and the matrix of N_w w.r.t this basis is of the form

$$[N_w]_{\mathbf{y}} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}, \tag{4.14}$$

where $W = {U_Q}^{-1}A$. Consequently we get that w.r.t the orthonormal basis $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$, the operator $N_w N_w^*$ has the matrix representation

$$\left[N_w N_w^*\right]_{\mathbf{y}} = \begin{pmatrix} W\overline{W}^t & 0 \\ 0 & 0 \end{pmatrix}.$$

Now by changing order of the orthonormal basis from $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ to $\tilde{\mathbf{y}} = (\mathbf{y}_1, \tilde{\mathbf{y}}_2)$ in an appropriate manner (as was explained in the previous section), we have

$$\left[N_w N_w^*\right]_{\tilde{\mathbf{y}}} = \begin{pmatrix} W\overline{W}^t & 0 \\ 0 & 0 \end{pmatrix}.$$

Comparing with equation (4.11), we infer that

$$W\overline{W}^t = \left(-\mathscr{K}_{\tilde{\gamma}}(w)\right)^{-1}. \tag{4.15}$$

So using eqn (4.14) we have matrix of $T|_{\mathscr{M}_w}$ w.r.t. the orthonormal basis $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ is of the form

$$\left[T|_{\mathscr{M}_w}\right]_{\mathbf{y}} = \begin{pmatrix} wI_n & W \\ 0 & wI_n \end{pmatrix},$$

where W satisfy the relation $W\overline{W}^t = \left(-\mathscr{K}_{\tilde{\gamma}}(w)\right)^{-1}$. Now it is straightforward to verify that for a rational function r whose poles are off $\overline{\Omega}^*$,

$$\left(r(T_{|\mathscr{M}_w})\right)_{\mathbf{y}} = \begin{pmatrix} r(w)I_n & r'(w)W \\ 0 & r(w)I_n \end{pmatrix}.$$

Since $\overline{\Omega}^*$ is a spectral set for T we have, $\sup\{\|r(T)\| : r \in \text{Rat}(\overline{\Omega}^*), \|r\|_\infty \leq 1, r(w) = 0\} \leq 1$. This gives us

$$4\pi^2 S_{\Omega^*}(w,w)^2 W\overline{W}^t \leq I_n. \tag{4.16}$$

where $S_{\Omega^*}(x,y)$ denote the Sżego kernel for the domain Ω^* which satisfy

$$2\pi S_{\Omega^*}(w,w) = \sup\{|r'(w)| : r \in \text{Rat}(\overline{\Omega}^*), \|r\|_\infty \leq 1, r(w) = 0\}.$$

From equation (4.15) and equation (4.16), we get

$$\mathscr{K}_{\tilde{\gamma}}(w) \leq -4\pi^2 S_{\Omega^*}(w,w)^2 I_n. \tag{4.17}$$

Note that when $n = 1$, that is, $T \in B_1(\Omega^*)$, then curvature function $\mathcal{K}_\sigma(w)$ is independent of the frame σ. Then we denote the curvature function by $\mathcal{K}_T(w)$. And from equation (4.17), we get our known curvature inequality

$$\mathcal{K}_T(w) \leq -4\pi^2 S_{\Omega^*}(w,w)^2 \text{ for all } w \in \Omega^*.$$

In general, for $T \in B_n(\Omega^*)$, the curvature matrix $\mathcal{K}_\sigma(w)$ w.r.t a frame σ is not independent of the choice of the frame. However, the transformation rule for curvature for a change of frame ensure that the eigenvalues of $\mathcal{K}_\sigma(w)$ are independent of the choice of a frame. Thus from equation (4.17), we have the following theorem:

Theorem 4.5. *Let Ω^* be a bounded domain in the complex plane $\mathbb{C}$ and T be an operator in $B_n(\Omega^*)$ admitting $\overline{\Omega}^*$ as a spectral set. Then each of the eigenvalues of the curvature $\mathcal{K}(w)$ of the bundle E_T is less or equal to $-4\pi^2 S_{\Omega^*}(w,w)^2$ for every $w \in \Omega^*$.*

Chapter 5

Module tensor Product and dilation

5.1 Preliminaries on Hilbert module

In this section, we discuss Hilbert modules in the Cowen-Douglas class and their tensor products. First we recall the definition of a Hilbert modules over a function algebra.

Definition 5.1 (Hilbert Module)**.** A Hilbert module $\mathscr{H}$ over a normed, unital, complex algebra $\mathscr{A}$ consists of a separable complex Hilbert space $\mathscr{H}$ together with a continuous map $(a,h) \mapsto a \cdot h$ from $\mathscr{A} \times \mathscr{H}$ to $\mathscr{H}$ satisfying the following conditions:

For $a, b \in \mathscr{A}$, $h, h_i \in \mathscr{H}$, and $\alpha, \beta \in \mathbb{C}$,

1. $1 \cdot h = h$,
2. $(ab) \cdot h = a \cdot (b \cdot h)$,
3. $(a+b) \cdot h = a \cdot h + b \cdot h$, and
4. $a \cdot (\alpha h_1 + \beta h_2) = \alpha(a \cdot h_1) + \beta(a \cdot h_2)$.

For a in $\mathscr{A}$, let $T_a : \mathscr{H} \mapsto \mathscr{H}$ denote the linear map $T_a(h) = a \cdot h$. If $\mathscr{H}$ is a Hilbert module over $\mathscr{A}$, then the continuity of the module action in the second variable ensures that T_a is bounded. However, the continuity of the module map in both the variables together with the principle of uniform boundedness leads to a slightly stronger conclusion in the Proposition given below.

Proposition 5.2. *[19, Proposition 1.3] Let $\mathscr{H}$ be a Hilbert module over a function algebra $\mathscr{A}$. Then there exists a constant K such that $\|T_a\| \leq K\|a\|$ for all $a \in \mathscr{A}$.*

The notion of a Hilbert module $\mathscr{H}$ over the algebra $\mathscr{A}$ and that of a continuous homomorphism of the algebra $\mathscr{A}$ into the algebra of bounded linear operators $\mathscr{L}(\mathscr{H})$ are in bijective correspondence. Indeed, if $\mathscr{H}$ is a Hilbert module over $\mathscr{A}$, then

$$\rho_{\mathscr{H}} : \mathscr{A} \mapsto \mathscr{L}(\mathscr{H}),\ \rho_{\mathscr{H}}(a) = T_a$$

is clearly a unital algebra homomorphism. Also, it is easy to verify that

$$\|\rho_{\mathscr{H}}\| = \inf\{K : \|T_a\| \leq K\|a\|,\ \ a \in \mathscr{A}\}.$$

Conversely, any continuous homomorphism $\rho : \mathscr{A} \mapsto \mathscr{L}(\mathscr{H})$ defines a Hilbert module via the action $(a, h) \mapsto \rho(a)h$. A Hilbert module $\mathscr{H}$ is said to be contractive if $\|\rho_{\mathscr{H}}\| \leq 1$.

Let Ω be a bounded, open and connected subset of $\mathbb{C}^m$. Let $\mathscr{H}$ be a complex separable Hilbert space. We shall assume that $\mathscr{H}$ is a Hilbert module over the function algebra $\mathscr{O}(\overline{\Omega})$, consisting of all those functions which are holomorphic in some neighbourhood of $\overline{\Omega}$, equipped with sup norm on $\overline{\Omega}$. Now we give a list of Hilbert modules and discuss several of their properties in detail.

Example 5.3. Let $\boldsymbol{T} := (T_1, \ldots, T_m)$ be an m-tuple of commuting jointly subnormal operator in $\mathscr{L}(\mathscr{H})$ with joint spectrum, in the sense of Taylor (cf. [37]), contained in $\overline{\Omega}$ and let $\boldsymbol{N} := (N_1, \ldots, N_m)$ be its minimal commuting normal extension to a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. Then Putinar has shown that the joint spectrum $\sigma(\boldsymbol{N})$ of $\boldsymbol{N}$ is contained in the joint spectrum $\sigma(\boldsymbol{T})$, of $\boldsymbol{T}$ (cf. [32]). This is the spectral inclusion theorem for commuting tuple of joint subnormal operators. For every $r \in \mathscr{O}(\overline{\Omega})$, we have $\|r(\boldsymbol{N})\|$ equals to $\sup\{|r(w)| : w \in \sigma(\boldsymbol{N})\}$. From the spectral inclusion theorem of Putinar, it follows that $\overline{\Omega}$ is a spectral set for the subnormal tuple of operators $\boldsymbol{T}$, that is,

$$\|r(\boldsymbol{T})\| \leq \|r(\boldsymbol{N})\| \leq \|r\|_\infty,\ \ r \in \mathscr{O}(\overline{\Omega}),$$

where $\|r\|_\infty = \sup\{|r(w)| : w \in \overline{\Omega}\}$. Thus $\mathscr{H}$ is a contractive Hilbert module over $\mathscr{O}(\overline{\Omega})$, where the module action is given by $r \cdot h = r(\boldsymbol{T})(h)$, for all $h \in \mathscr{H}$.

Example 5.4. Let $w = (w_1, \ldots, w_m)$ be an arbitrary but fixed point in Ω. For a non zero vector $a = (a_1, \ldots, a_m)$ in $\mathbb{C}^m$, let $\mathbb{C}^2_w(a)$ be the Hilbert module over the function algebra $\mathscr{O}(\overline{\Omega})$ with the module action given by

$$r \cdot v = \begin{pmatrix} r(w) & 0 \\ (\triangledown r(w).a) & r(w) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad r \in \mathscr{O}(\overline{\Omega}), v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2,$$

where $(\triangledown r(w).a) = a_1 \frac{\partial}{\partial z_1} r(w) + \cdots + a_m \frac{\partial}{\partial z_m} r(w)$.

This module action on $\mathbb{C}^2_w(a)$ is induced by the following commuting m-tuple of 2×2 matrices whose joint spectrum equal to $\{w\}$:

$$\boldsymbol{T} := \Big(\begin{pmatrix} w_1 & 0 \\ a_1 & w_1 \end{pmatrix}, \ldots, \begin{pmatrix} w_m & 0 \\ a_m & w_m \end{pmatrix} \Big).$$

So, we have, $r\cdot v = r(\boldsymbol{T})(v)$, for all v in $\mathbb{C}^2_w(a)$. Following Lemma gives a complete characterization of contractivity of the module $\mathbb{C}^2_w(a)$ in terms of the Carathéodory norm $C_{\Omega,w}(a)$, which is defined to be the solution to the following extremal problem, that is,

$$C_{\Omega,w}(a) := \sup\Big\{\tfrac{|(\nabla r(w).a)|}{1-|r(w)|^2} : r\in\mathscr{O}(\overline{\Omega}), \|r\|_\infty \le 1\Big\}.$$

Let r be a non constant function in $\mathscr{O}(\overline{\Omega})$ with $\|r\|_\infty \le 1$. Let $\varphi_{r(w)}$ be the automorphism of unit disk $\mathbb{D}$, defined by $\varphi_{r(w)}(z) = \frac{z-r(w)}{1-\overline{r(w)}z}$, $z\in\mathbb{D}$. Note that $\varphi_{r(w)}\circ r$ is a function in $\mathscr{O}(\overline{\Omega})$ with $\|\varphi_{r(w)}\circ r\|_\infty \le 1$ and $(\varphi_{r(w)}\circ r)(w) = 0$. It is easy to verify that

$$|(\nabla(\varphi_{r(w)}\circ r)(w).a)| = \tfrac{|(\nabla r(w).a)|}{1-|r(w)|^2}.$$

Now it is evident that the Carathéodory norm, $C_{\Omega,w}(a)$, is also equal to the solution of the following extremal problem:

$$C_{\Omega,w}(a) := \sup\{|(\nabla f(w).a)| : f\in\mathscr{O}(\overline{\Omega}), f(w)=0, \|f\|_\infty \le 1\}. \tag{5.1}$$

We state and give the easy proof of a lemma (cf. [31, Lemma 4.1]) which is useful in many of our computations which follow.

Lemma 5.5. *$\mathbb{C}^2_w(a)$ is a contractive Hilbert module if and only if $C_{\Omega,w}(a) \le 1$.*

Proof. For any two complex numbers a, c with $|a|, |c| \le 1$, the 2×2 matrix $X = \left(\begin{smallmatrix} a & 0 \\ b & c\end{smallmatrix}\right)$ is contractive if and only if $\det(I - X^*X) > 0$, that is, $|b|^2 \le (1-|a|^2)(1-|c|^2)$. Now it follows that the Hilbert module $\mathbb{C}^2_w(a)$ is contractive if and only if $|(\nabla r(w).a)| \le (1-|r(w)|^2)$, or equivalently,

$$\tfrac{|(\nabla r(w).a)|}{1-|r(w)|^2} \le 1,\ r\in\mathscr{O}(\overline{\Omega}),\ \|r\|_\infty \le 1.$$

Hence we conclude that the module $\mathbb{C}^2_w(a)$ is contractive if and only if $C_{\Omega,w}(a)\le 1$. $\square$

Example 5.6. Let $w_1 = (w_1^1, \cdots, w_m^1)$ and $w_2 = (w_1^2,\cdots,w_m^2)$ be two distinct arbitrary but fixed points in Ω. For a non zero scalar t in $\mathbb{C}$, let $\mathbb{C}^2_{w_1,w_2}(t(w_2-w_1))$ be the Hilbert module over the function algebra $\mathscr{O}(\overline{\Omega})$ with the module action given by

$$r\cdot v = \begin{pmatrix} r(w_1) & 0 \\ t(r(w_2)-r(w_1)) & r(w_2)\end{pmatrix}\begin{pmatrix} v_1 \\ v_2\end{pmatrix}, \quad r\in\mathscr{O}(\overline{\Omega}), v = \begin{pmatrix} v_1 \\ v_2\end{pmatrix}\in\mathbb{C}^2.$$

As before, this module action is induced by the following m-tuple $\boldsymbol{T}$ of commuting 2×2 matrices with joint spectrum equal to $\{w_1, w_2\}$:

$$\boldsymbol{T} := \left(\begin{pmatrix} w_1^1 & 0 \\ t(w_1^2 - w_1^1) & w_1^2 \end{pmatrix}, \ldots, \begin{pmatrix} w_m^1 & 0 \\ t(w_m^2 - w_m^1) & w_m^2 \end{pmatrix} \right)$$

and we have $r \cdot v = r(\boldsymbol{T})(v)$. The following lemma gives a criterion for the contractivity of the Hilbert modules $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ in terms of the möbius distance

$$m_\Omega(w_1, w_2) := \sup\left\{ \frac{|r(w_1) - r(w_2)|}{|1 - r(w_1)\overline{r(w_2)}|} : r \in \mathscr{O}(\overline{\Omega}), \|r\|_\infty \le 1 \right\}.$$

Using the Schwarz-Pick lemma, we have $m_{\mathbb{D}}(a,b) = \frac{|a-b|}{|1-\bar{a}b|}$, for every $a, b \in \mathbb{D}$. Schwarz-Pick lemma also tells us that $m_{\mathbb{D}}(\varphi(a), \varphi(b)) = m_{\mathbb{D}}(a,b)$ for every automorphism φ of unit disk $\mathbb{D}$ and $a, b \in \mathbb{D}$. Let r be a non constant function in $\mathscr{O}(\overline{\Omega})$ with $\|r\|_\infty \le 1$. Let $\varphi_{r(w_2)}$ be the automorphism of unit disk $\mathbb{D}$ which takes $r(w_2)$ to 0. Note that $\varphi_{r(w_2)} \circ r$ is a function in $\mathscr{O}(\overline{\Omega})$ with $\|\varphi_{r(w_2)} \circ r\|_\infty \le 1$ and $(\varphi_{r(w_2)} \circ r)(w_2) = 0$. From Schwarz-Pick lemma we have that

$$\frac{|r(w_1) - r(w_2)|}{|1 - r(w_1)\overline{r(w_2)}|} = \frac{|(\varphi_{r(w_2)} \circ r)(w_1) - (\varphi_{r(w_2)} \circ r)(w_2)|}{|1 - (\varphi_{r(w_2)} \circ r)(w_1)\overline{(\varphi_{r(w_2)} \circ r)(w_2)}|} = |(\varphi_{r(w_2)} \circ r)(w_1)|.$$

Now it is evident that the möbius distance, $m_\Omega(w_1, w_2)$, between the points w_1 and w_2 in Ω is also equal to the solution of the following extremal problem:

$$m_\Omega(w_1, w_2) := \sup\left\{ |f(w_1)| : f \in \mathscr{O}(\overline{\Omega}), \|f\|_\infty \le 1, f(w_2) = 0 \right\}. \tag{5.2}$$

Lemma 5.7. *The Hilbert module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ over the function algebra $\mathscr{O}(\overline{\Omega})$ is contractive if and only if $m_\Omega(w_1, w_2) \le \frac{1}{\sqrt{1+|t|^2}}$.*

Proof. We have seen earlier that for two complex number a, c with $|a|, |c| \le 1$, a scalar matrix $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ is contraction if and only if $|b|^2 \le (1-|a|^2)(1-|c|^2)$. Now it follows that the Hilbert module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ is contractive if and only if $|t(r(w_2) - r(w_1)|^2 \le (1 - |r(w_1)|^2)(1 - |r(w_2)|^2)$, or equivalently,

$$\frac{|r(w_1)-r(w_2)|}{|1-r(w_1)\overline{r(w_2)}|} \le \frac{1}{\sqrt{1+|t|^2}}, \; r \in \mathscr{O}(\overline{\Omega}) \; \|r\|_\infty \le 1.$$

Hence we conclude that the Hilbert module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ is contractive if and only if $(m_\Omega(w_1, w_2)) \le \frac{1}{\sqrt{1+|t|^2}}$. □

Definition 5.8 (Submodule)**.** Let $\mathscr{H}$ be a Hilbert module over a function algebra $\mathscr{A}$ and $\rho_{\mathscr{H}}$ be the associated homomorphism. A closed subspace $\mathscr{S}$ of $\mathscr{H}$ is said to be a submodule of $\mathscr{H}$ provided $a \cdot s \in \mathscr{S}$ for every $a \in \mathscr{A}$ and $s \in \mathscr{S}$. In other words, $\mathscr{S}$ is a invariant subspace for the operator $\rho_{\mathscr{H}}(a)$ for every $a \in \mathscr{A}$. Naturally, $\mathscr{S}$ is a Hilbert module whose associated homomorphism $\rho_{\mathscr{S}}$ is given by the restriction $\rho_{\mathscr{H}}(a)_{|\mathscr{S}}$, that is, $\rho_{\mathscr{S}}(a) = \rho_{\mathscr{H}}(a)_{|\mathscr{S}}$, $a \in \mathscr{A}$.

Definition 5.9 (Quotient module)**.** Let $\mathscr{H}$ be a Hilbert module over a function algebra $\mathscr{A}$ and $\rho_{\mathscr{H}}$ be the associated homomorphism. Let $\mathscr{S}$ be a Hilbert submodule of $\mathscr{H}$. Then the subspace $\mathscr{S}^{\perp}$ can be made a Hilbert module over $\mathscr{A}$ in a natural manner. Indeed, the module action on $\mathscr{S}^{\perp}$ is given by $(a, v) \mapsto P_{\mathscr{S}^{\perp}}(a \cdot v)$, for $a \in \mathscr{A}$ and $v \in \mathscr{S}^{\perp}$, where $P_{\mathscr{S}^{\perp}}$ is the orthogonal projection from $\mathscr{H}$ onto $\mathscr{S}^{\perp}$. In other words,

$$0 \longrightarrow \mathscr{S} \xrightarrow{i} \mathscr{H} \xrightarrow{P_{\mathscr{S}^{\perp}}} \mathscr{S}^{\perp} \longrightarrow 0$$

is a short exact sequence of Hilbert modules over the algebra $\mathscr{A}$. Thus $\mathscr{S}^{\perp}$ is the quotient module. Note that the associated homomorphism $\rho_{\mathscr{S}^{\perp}}$ for the quotient module $\mathscr{S}^{\perp}$ is given by the compression of $\rho_{\mathscr{H}}(a)$ into $\mathscr{S}^{\perp}$, that is, $\rho_{\mathscr{S}^{\perp}}(a) = P_{\mathscr{S}^{\perp}}(\rho_{\mathscr{H}}(a))_{|\mathscr{S}^{\perp}}$ for every $a \in \mathscr{A}$.

Remark 5.10. Let $\mathscr{M}$ be a closed subspace of a Hilbert module $\mathscr{H}$ over $\mathscr{A}$. In general, the compression of the module action to $\mathscr{M}$, that is, $(a, m) \mapsto P_{\mathscr{M}}(a \cdot m)$ for $a \in \mathscr{A}$ and $m \in \mathscr{M}$ need not define a module action. The associativity required of the module product fails. It defines a module action on $\mathscr{M}$ if and only if $P_{\mathscr{M}}(a_1 a_2 \cdot m) = P_{\mathscr{M}}(a_1 \cdot (P_{\mathscr{M}}(a_2 \cdot m)))$ for every $a_1, a_2 \in \mathscr{A}$ and $m \in \mathscr{M}$. Such modules are often called semi-submodule of the Hilbert module $\mathscr{H}$. In [35, Lemma 0], Sarason has provided a complete characterization of a semi-submodule.

Theorem 5.11 (Sarason). *Let $\mathscr{H}$ be a Hilbert module over $\mathscr{A}$ and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. For $a \in \mathscr{A}$, the map $(a, m) \mapsto P_{\mathscr{M}}(a \cdot m)$, $m \in \mathscr{M}$, defines a module action on $\mathscr{M}$ over the algebra $\mathscr{A}$ if and only if there exist two submodules $\mathscr{F}$ and $\mathscr{G}$ of the Hilbert module $\mathscr{H}$ such that $\mathscr{F} \subseteq \mathscr{G}$ and $\mathscr{M} = \mathscr{G} \ominus \mathscr{F}$.*

Definition 5.12 (Equivalence of Hilbert module)**.** Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert module over a function algebra $\mathscr{A}$. Let $\rho_{\mathscr{H}}$ and $\rho_{\mathscr{K}}$ be the associated homomorphism from $\mathscr{A}$ into $\mathscr{L}(\mathscr{H})$ and $\mathscr{L}(\mathscr{K})$ respectively. A bounded linear map $X : \mathscr{H} \mapsto \mathscr{K}$ is said to be a module map if $X\rho_{\mathscr{H}}(a) = \rho_{\mathscr{K}}(a)X$ for every $a \in \mathscr{A}$. The modules $\mathscr{H}$ and $\mathscr{K}$ are said to be similar if there exist a invertible module map from $\mathscr{H}$ onto $\mathscr{K}$ and the modules are said to be isomorphic if there exists a unitary module map from $\mathscr{H}$ onto $\mathscr{K}$.

Let w, w' be two points in Ω and a, b be two vectors in $\mathbb{C}^m$. Consider the Hilbert module $\mathbb{C}^2_w(a)$ and $\mathbb{C}^2_{w'}(b)$ over the algebra $\mathscr{O}(\overline{\Omega})$, as described in example 5.4. It is easy to verify that the modules $\mathbb{C}^2_w(a)$ and $\mathbb{C}^2_{w'}(b)$ are isomorphic if and only if $w = w'$ and $a = \lambda b$, for some complex number λ in the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Similarly, for the Hilbert modules $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ and $\mathbb{C}^2_{z_1,z_2}(s(z_2 - z_1))$, as described in example 5.6, are isomorphic if and only if $(w_1, w_2) = (z_1, z_2)$ and $t = \lambda s$, for some $\lambda \in \mathbb{T}$.

Let $\mathscr{A}$ be a normed, unital, complex function algebra and $M_{\mathscr{A}}$ be its maximal ideal space. Let $\mathscr{C}(M_{\mathscr{A}})$ be the set of all continuous function on $M_{\mathscr{A}}$, where $M_{\mathscr{A}}$ is equipped with weak star topology. We have that $\mathscr{A}$ is a sub-algebra of $\mathscr{C}(M_{\mathscr{A}})$ via the Gelfand transform (cf. [12, Ch.7]). The smallest closed subset of $M_{\mathscr{A}}$ on which each functions in $\mathscr{A}$ achieves its maximum modulus, is called the Ŝilov boundary for $\mathscr{A}$ and is denoted by $\partial\mathscr{A}$. Existence of such boundary is well known (see [21, pg. 9]). Thus, we may regard $\mathscr{A}$ as a sub-algebra of the function algebra $\mathscr{C}(\partial\mathscr{A})$.

Definition 5.13 (Dilation)**.** Let $\mathscr{H}$ be a Hilbert module over a function algebra $\mathscr{A}$ and $\rho_{\mathscr{H}}$ be the associated homomorphism. The module $\mathscr{H}$ is said to have a $\partial\mathscr{A}$ dilation if there exist a contractive Hilbert module $\mathscr{K} \supseteq \mathscr{H}$ over $\mathscr{C}(\partial\mathscr{A})$ such that $P_{\mathscr{H}}\rho_{\mathscr{K}}(a)_{|\mathscr{H}} = \rho_{\mathscr{H}}(a)$ for all $a \in \mathscr{A}$.

It is now evident (see Theorem 5.11) that the existence of a $\partial\mathscr{A}$ dilation is equivalent to the existence of a pair of nested submodules of $\mathscr{K}$ over $\mathscr{A}$. The notion of a module which admits a $\partial\mathscr{A}$ dilation and the complete contractivity of the homomorphism determined by the module action are intimately related. First, recall the definition of complete contractivity.

Let $\mathscr{A}$ be a normed unital function algebra and $\mathscr{M}_n$ be the C*algebra of $n \times n$ complex matrices. Since $\mathscr{A}$ is a sub-algebra of the C* algebra $\mathscr{C}(\partial\mathscr{A})$, we regard $\mathscr{A} \otimes \mathscr{M}_n$ as a sub-algebra of the C* algebra $\mathscr{C}(\partial\mathscr{A}) \otimes \mathscr{M}_n$. Let $\mathscr{H}$ be a Hilbert module over $\mathscr{A}$ and $\rho_{\mathscr{H}} : \mathscr{A} \mapsto \mathscr{L}(\mathscr{H})$ be the associated homomorphism. The module $\mathscr{H}$ is said to be completely contractive over $\mathscr{A}$ if the map $\rho \otimes I_n : \mathscr{A} \otimes \mathscr{M}_n \mapsto \mathscr{L}(\mathscr{H}) \otimes \mathscr{M}_n$ is contractive for every $n \in \mathbb{N}$.

A deep theorem due to Arveson states that a Hilbert module $\mathscr{H}$ over $\mathscr{A}$ admits a $\partial\mathscr{A}$ dilation if and only if the module $\mathscr{H}$ is completely contractive over $\mathscr{A}$ (cf. [5, Theorem 1.2.2]). A contractive module $\mathscr{H}$ over a function algebra $\mathscr{A}$ is not necessarily completely contractive [27]. However, the modules described in Examples 5.4 and 5.6 are completely contractive if and only if it is contractive [4, Theorem 1.9]. In fact, Agler [4, Proposition 3.5] has shown that the module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$ is completely contractive if and only if it is contractive. Following this he remarked that a similar statement can be made for the modules $\mathbb{C}^2_w(a)$ using a limiting argument. However, it is possible to show that the module $\mathbb{C}^2_w(a)$ is contractive if and only if it is completely contractive directly. We provide a short proof here.

Let us assume that the Hilbert module is $\mathbb{C}^2_w(a)$ is contractive. By Lemma 5.5, we have $C_{\Omega,w}(a) \le 1$. Recall that the module action on $\mathbb{C}^2_w(a)$ is induced by the following commuting m-tuple of 2×2 matrices whose joint spectrum equal to $\{w\}$:

$$\boldsymbol{T} := \left(\begin{pmatrix} w_1 & 0 \\ a_1 & w_1 \end{pmatrix}, \ldots, \begin{pmatrix} w_m & 0 \\ a_m & w_m \end{pmatrix} \right).$$

So, we have, $r \cdot v = r(\boldsymbol{T})(v)$, for all $v \in \mathbb{C}^2_w(a)$ and for all $r \in \mathscr{O}(\overline{\Omega})$. Let $(\mathscr{M}_k)_1$ be the open unit ball with respect to the operator norm in $k \times k$ matrices and $F : \overline{\Omega} \to (\mathscr{M}_k)_1$ be a holomorphic function. We have

$$F(\boldsymbol{T}) = \begin{pmatrix} F(w) & 0 \\ DF(w)(a) & F(w) \end{pmatrix},$$

where $DF(w)(a) = a_1\frac{\partial}{\partial z_1}F(w) + \cdots + a_m\frac{\partial}{\partial z_m}F(w)$. So, the module $\mathbb{C}^2_w(a)$ is completely contractive if and only if $\|F(\boldsymbol{T})\|_{Op} \leq 1$, for every holomorphic $F : \overline{\Omega} \to (\mathcal{M}_k)_1$ and $k \in \mathbb{N}$. Using zero lemma (cf. [28, Lemma 1.6]), we get that the module $\mathbb{C}^2_w(a)$ is completely contractive if and only if $\|DF(w)(a)\|_{Op} \leq 1$, for every holomorphic $F : \overline{\Omega} \to (\mathcal{M}_k)_1$ satisfying $F(w) = 0$ and $k \in \mathbb{N}$. The holomorphic function F is easily seen to be (Carathéodory) norm decreasing and the Carathéodory norm at 0 of $(\mathcal{M}_k)_1$ is the operator norm. Thus for any holomorphic map $F : \overline{\Omega} \to (\mathcal{M}_k)_1$ such that $F(w) = 0$, we have that

$$\|DF(w)(a)\|_{\mathrm{Op}} = C_{((\mathcal{M}_k)_1,0)}(DF(w)(a)) \leq C_{\Omega,w}(a) \leq 1.$$

This completes the proof.

5.2 Module tensor product

Definition 5.14 (Module tensor product)**.** Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert modules over a function algebra $\mathcal{A}$. There are two natural ways to make $\mathcal{H} \otimes \mathcal{K}$ a Hilbert module. One can define module actions via $a \cdot (h \otimes k) \mapsto (a \cdot h) \otimes k$ or $a \cdot (h \otimes k) \mapsto h \otimes (a \cdot k)$ These are known as the left and right module tensor products of $\mathcal{H}$ and $\mathcal{K}$ and these are denoted by ${}_{\mathcal{A}}(\mathcal{H} \otimes \mathcal{K})$ and $(\mathcal{H} \otimes \mathcal{K})_{\mathcal{A}}$ respectively. Consider the closed subspace $\mathcal{N}$ of $\mathcal{H} \otimes \mathcal{K}$ generated by vectors of the form $(a \cdot h) \otimes k - h \otimes (a \cdot k)$ for $h \in \mathcal{H}$ and $k \in \mathcal{K}$. Clearly $\mathcal{N}$ is a submodule of both the left and right module tensor products. Now consider the quotient module ${}_{\mathcal{A}}(\mathcal{H} \otimes \mathcal{K})/\mathcal{N}$ and $(\mathcal{H} \otimes \mathcal{K})_{\mathcal{A}}/\mathcal{N}$. As Hilbert space both these modules can be identified as $\mathcal{N}^{\perp}$ with the module action given by the compression of the left and the right module action to $\mathcal{N}^{\perp}$, respectively. Since $P_{\mathcal{N}^{\perp}}((a \cdot h) \otimes k) = P_{\mathcal{N}^{\perp}}(h \otimes (a \cdot k))$, it follows that these quotient modules are isomorphic. Such a quotient module is called the module tensor product of $\mathcal{H}$ and $\mathcal{K}$ over the function algebra $\mathcal{A}$ and we let $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{K}$ denote this module tensor product.

Let Ω be a bounded domain in $\mathbb{C}^m$. Set $\Omega^* = \{\bar{z} \mid z \in \Omega\}$, which is again a bounded domain in $\mathbb{C}^m$. Let $\mathcal{H}_K$ be a Hilbert space consisting of holomorphic function on Ω possessing a reproducing kernel K. Thus $K : \Omega \times \Omega \to \mathbb{C}$ is holomorphic in the first variable and anti-holomorphic in the second. Let $\boldsymbol{M} = (M_{z_1}, \ldots, M_{z_m})$ be the commuting m-tuple of multiplication by the coordinate functions on $\mathcal{H}_K$, which are assumed to be bounded. Let $\boldsymbol{M}^*$ denote the commuting m-tuple $(M^*_{z_1}, \ldots, M^*_{z_m})$. Recall that any commuting m-tuple operators $\boldsymbol{T} = (T_1, \ldots, T_m)$ in the Cowen-Douglas class $B_1(\Omega^*)$ is jointly unitarily equivalent to the m-tuple $\boldsymbol{M}^*$ on some reproducing kernel Hilbert space $\mathcal{H}_K$, (cf. [16]). We will assume, without loss of generality, that an operator tuple $\boldsymbol{T}$ in $B_1(\Omega^*)$ has been realized as above.

We further assume that $\overline{\Omega}$ is a $c-$ spectral set for the operator tuple $\boldsymbol{M}$, that is, $\sigma(\boldsymbol{M}) \subseteq \overline{\Omega}$ and

$$\|r(\boldsymbol{M})\| \leq c\|r\|_{\infty},\ r \in \mathcal{O}(\overline{\Omega}),$$

for some constant $c > 0$. Thus the module action

$$r \cdot h = r(\boldsymbol{M})(h), \;\; r \in \mathcal{O}(\overline{\Omega}), h \in \mathcal{H}_K.$$

defined by the m- tuple of operators $\boldsymbol{M}$ makes $\mathcal{H}_K$ a Hilbert module over $\mathcal{O}(\overline{\Omega})$.

Let $w = (w_1, \cdots, w_m)$ be an arbitrary but fixed point in Ω. For a non-zero vector $a = (a_1, \ldots, a_m)$ in $\mathbb{C}^m$, from example 5.4, we have that $\mathbb{C}^2_w(a)$ is the Hilbert module over the function algebra $\mathcal{O}(\overline{\Omega})$. Now we want to find the module tensor product $\mathcal{H}_K \otimes_{\mathcal{O}(\overline{\Omega})} \mathbb{C}^2_w(a)$, which is the orthocomplement of the subspace $\mathcal{N}$ of $\mathcal{H}_K \otimes \mathbb{C}^2$, where

$$\mathcal{N} = \vee\{r \cdot h \otimes v - h \otimes r \cdot v \mid r \in \mathcal{O}(\overline{\Omega}), h \in \mathcal{H}_K, v \in \mathbb{C}^2\}.$$

The module action on $\mathcal{N}^\perp$ is induced by the compression $P_{\mathcal{N}^\perp}(\boldsymbol{M} \otimes I)_{|\mathcal{N}^\perp}$. Now we will find an orthonormal basis for $\mathcal{N}^\perp$ and the matrix representation of $P_{\mathcal{N}^\perp}(\boldsymbol{M} \otimes I)_{|\mathcal{N}^\perp}$ w.r.t that basis. Let $N(w)$ and $N^2(w)$ be the subspaces of $\mathcal{H}(K)$ defined by

$$\begin{aligned} N(w) &= \bigcap_{k=1}^{m} \operatorname{Ker}(M_k^* - \bar{w}_k) = \mathbb{C}[\gamma(w)] \text{ and} \\ N^2(w) &= \bigcap_{k,l=1}^{m} \operatorname{Ker}(M_k^* - \bar{w}_k)(M_l^* - \bar{w}_l) = \vee\{\gamma(w), \bar{\partial}_1\gamma(w), \ldots, \bar{\partial}_m\gamma(w)\}, \end{aligned}$$

where $\gamma(w) = K(\cdot, w)$ and $\bar{\partial}_i\gamma(w) = \frac{\partial}{\partial \bar{w}_i}K(\cdot, w)$. It is easy to see that, $(f \otimes e_1 + g \otimes e_2) \in \mathcal{N}^\perp$ if and only if $\langle (r - r(w)).h, g\rangle_{\mathcal{H}_K} = 0$ and $\langle (r - r(w)).h, f\rangle_{\mathcal{H}_K} = (\nabla r(w).a)\langle h, g\rangle_{\mathcal{H}_K}$ for every $h \in \mathcal{H}_K$ and $r \in \mathcal{A}(\overline{\Omega})$. It then follows that $g \in N(w)$ and $f \in N^2(w)$. Also, $(f \otimes e_1 + g \otimes e_2) \in \mathcal{N}^\perp$ if and only if

$$g = c_1\gamma(w) \;\text{ and }\; g = c_2\gamma(w) + c(\bar{a}_1\bar{\partial}_1\gamma(w) + \cdots + \bar{a}_m\bar{\partial}_m\gamma(w)).$$

for some scalars c_1 and c_2. Consequently we have that

$$\left\{\begin{pmatrix}\gamma(w)\\ 0\end{pmatrix}, \begin{pmatrix}\bar{a}_1\bar{\partial}_1\gamma(w) + \cdots + \bar{a}_m\bar{\partial}_m\gamma(w))\\ \gamma(w)\end{pmatrix}\right\}$$

is a basis for $\mathcal{N}^\perp$. For brevity of notation let's denote the vector $(\bar{a}_1\bar{\partial}_1\gamma(w) + \cdots + \bar{a}_m\bar{\partial}_m\gamma(w))$ in $\mathcal{H}_K$ by $\bar{\partial}_a\gamma(w)$. Since $(M_l^* - \bar{w}_l)(\gamma(w)) = 0$ and $(M_l^* - \bar{w}_l)(\bar{\partial}_i\gamma(w)) = \delta_{i,l}(\gamma(w))$, the matrix representation of $(M_l^* \otimes I)_{|\mathcal{N}^\perp}$ is of the form

$$[(M_l^* \otimes I)_{|\mathcal{N}^\perp}] = \begin{pmatrix}\bar{w}_l & \bar{a}_l\\ 0 & \bar{w}_l\end{pmatrix}.$$

Now we find the matrix representation of $(M_l^* \otimes I)_{|\mathcal{N}^\perp}$ w.r.t the orthonormal basis obtained from this pair of linearly independent vectors using the Gram-Schmidt orthogonalization. We first prove a useful Lemma.

Lemma 5.15. *Let $\beta_v = \{v_1, v_2\}$ be a basis of a inner product space V and A be a linear map in $\mathscr{L}(V)$ whose matrix representation w.r.t the basis β_v is of the form*

$$\begin{pmatrix} \lambda & \mu \\ 0 & \eta \end{pmatrix}.$$

Let $\beta_u = \{u_1, u_2\}$ be the orthonormal basis obtained from β_v by applying Gram-Schmidt orthonormalization. Then w.r.t the basis β_u, the matrix of A will be of the form

$$\begin{pmatrix} \lambda & \frac{(\eta-\lambda)\langle v_2, v_1\rangle + \mu \|v_1\|^2}{\sqrt{\|v_1\|^2\|v_2\|^2 - |\langle v_1, v_2\rangle|^2}} \\ 0 & \eta \end{pmatrix}.$$

Proof. The proof is a straightforward computation. Note that the basis β_v can be transformed to the basis β_u by the transition matrix $P : (u_1, u_2) = (v_1, v_2)P$, where

$$P = \begin{pmatrix} \frac{1}{\|v_1\|} & \frac{-\langle v_2, v_1\rangle}{\|v_1\|\sqrt{\|v_1\|^2\|v_2\|^2 - |\langle v_1, v_2\rangle|^2}} \\ 0 & \frac{\|v_1\|}{\sqrt{\|v_1\|^2\|v_2\|^2 - |\langle v_1, v_2\rangle|^2}} \end{pmatrix}.$$

Then the matrix $[A]_{\beta_u}$ of A w.r.t the basis β_u is easily seen to be $P^{-1}[A]_{\beta_v}P$. An easy computation shows that it is of the form claimed in the Lemma. □

With respect to the orthonormal basis obtained in the preceding Lemma, matrix representation of the operator $(M_{z_l}^* \otimes I)_{|\mathcal{N}^\perp}$ takes the form

$$[(M_{z_l}^* \otimes I)_{|\mathcal{N}^\perp}] = \begin{pmatrix} \bar{w}_l & \frac{\bar{a}_l}{\sqrt{1 - \langle \mathcal{K}_K(\bar{w})a, a\rangle}} \\ 0 & \bar{w}_l \end{pmatrix},$$

where

$$\langle \mathcal{K}_K(\bar{w})a, a\rangle = -\frac{\|\gamma(w)\|^2\|\bar{\partial}_a\gamma(w)\|^2 - |\langle\gamma(w), \bar{\partial}_a\gamma(w)\rangle|^2}{\|\gamma(w)\|^4}.$$

We pause to point out that $\mathcal{K}_K(\bar{w})$, $\bar{w} \in \Omega^*$, is the curvature of the operator $\boldsymbol{M}^*$ acting on the Hilbert space $\mathscr{H}_K$. It is the $(1,1)-$ form

$$\sum_{i,j=1}^{m} \mathcal{K}_{i,j}(\bar{w})\, dz_i \wedge d\bar{z}_j,$$

where

$$\begin{aligned} \mathcal{K}_{i,j}(\bar{w}) &= -\big(\partial_j\bar{\partial}_i \log K\big)(w, w) \\ &= -\frac{\|\gamma(w)\|^2\langle\bar{\partial}_i\gamma(w), \bar{\partial}_j\gamma(w)\rangle - \langle\bar{\partial}_i\gamma(w), \gamma(w)\rangle\langle\gamma(w), \bar{\partial}_j\gamma(w)\rangle}{\|\gamma(w)\|^4}, \quad w \in \Omega. \end{aligned}$$

Consequently, we have that

$$[P_{\mathcal{N}^\perp}\big(M_{z_l}\otimes I\big)_{|\mathcal{N}^\perp}] = \begin{pmatrix} w_l & 0 \\ \frac{a_l}{\sqrt{1-\langle \mathcal{K}_K(\bar{w})a,a\rangle}} & w_l \end{pmatrix}. \tag{5.3}$$

Hence we have proved the following.

Proposition 5.16. *The module tensor product $\mathcal{H}_K \otimes_{\mathcal{O}(\overline{\Omega})} \mathbb{C}^2_w(a)$ is isomorphic to $\mathbb{C}^2_w(\hat{a})$, where $\hat{a}$ is the vector given by $\hat{a} = \frac{a}{\sqrt{1-\langle \mathcal{K}_K(\bar{w})a,a\rangle}}$.*

Remark 5.17. Notice that for any non zero λ in $\mathbb{C}$, we have

$$\widehat{(\lambda a)} = \frac{\lambda}{\sqrt{1-|\lambda|^2\langle \mathcal{K}_K(\bar{w})a,a\rangle}}\,a.$$

Clearly the set $\{\widehat{(\lambda a)} : \lambda \in \mathbb{C}, \lambda \neq 0\}$ is equal to the set $\{ta : |t| \in \big(0, \frac{1}{\sqrt{-\langle \mathcal{K}_K(\bar{w})a,a\rangle}}\big)\}$. If we assume that the tuple of operator $\boldsymbol{M}$ acting on $\mathcal{H}_K$ is jointly subnormal with normal spectrum in the Ŝilov boundary of $\overline{\Omega}$, then for every non zero vector $a \in \mathbb{C}^n$, it follows from Sarason's Theorem 5.11 that the module tensor product $\mathcal{H}_K \otimes_{\mathcal{O}(\overline{\Omega})} \mathbb{C}^2_w(a) = \mathbb{C}^2_w(\hat{a})$ admits a boundary dilation. So, for every non zero vector $a \in \mathbb{C}^n$, the Hilbert module $\mathbb{C}^2_w(ta)$, where $|t| \in \big(0, \frac{1}{\sqrt{-\langle \mathcal{K}_K(\bar{w})a,a\rangle}}\big)$, admits a boundary dilation if the operator tuple $\boldsymbol{M}$, acting on $\mathcal{H}_K$, is jointly subnormal with normal spectrum in Ŝilov boundary of $\overline{\Omega}$.

Let $w_1 = (w^1_1,\ldots,w^1_m)$ and $w_2 = (w^2_1,\ldots,w^2_m)$ be two distinct arbitrary but fixed point in Ω. For a non zero scalar t in $\mathbb{C}$, from example 5.6, we have $\mathbb{C}^2_{w_1,w_2}(t(w_2-w_1))$ is the Hilbert module over the function algebra $\mathcal{O}(\overline{\Omega})$. The module tensor product $\mathcal{H}_K \otimes_{\mathcal{O}(\overline{\Omega})} \mathbb{C}^2_{w_1,w_2}(t(w_2-w_1))$, in this case, is found by methods similar to the ones used for the case $w_1 = w_2$. Therefore, we shall not repeat all the details. Let

$$\mathcal{N} = \vee\{r\cdot h\otimes v - h\otimes r\cdot v \mid r \in \mathcal{O}(\overline{\Omega}), h \in \mathcal{H}_K, v \in \mathbb{C}^2\} \subseteq \mathcal{H}_K \otimes \mathbb{C}^2.$$

As before, the compression of the operator tuple $P_{\mathcal{N}^\perp}(\boldsymbol{M}\otimes I)|_{\mathcal{N}^\perp}$ defines the required module tensor product. We find an orthonormal basis for $\mathcal{N}^\perp$ and the matrix representation for this compression w.r.t this basis. Let $N(w_2)$ and $N(w_1,w_2)$ be the subspaces of $\mathcal{H}_K$ defined by

$$N(w_2) = \bigcap_{k=1}^m \mathrm{Ker}(M_k^* - \bar{w}_k^2) = \vee\{K(\cdot,w_2)\}, \text{ and}$$
$$N(w_1,w_2) = \bigcap_{j,k=1}^m \mathrm{Ker}(M_j^* - \bar{w}_j{}^1)(M_k^* - \bar{w}_k^2) = \vee\{K(\cdot,w_1),K(\cdot,w_2)\}.$$

It is easy to see that, $(f\otimes e_1 + g\otimes e_2) \in \mathcal{N}^\perp$ if and only if $\langle (r-r(w_2)).h, g\rangle_{\mathcal{H}_K} = 0$ and $\langle (r-r(w_1)).h, f\rangle_{\mathcal{H}_K} = a(r(w_2)-r(w_1))\langle h,f\rangle_{\mathcal{H}_K}$ for every $h \in \mathcal{H}_K$ and $r \in \mathcal{A}(\overline{\Omega})$. It follows that $g \in N(w_2)$ and $f \in N(w_1,w_2)$. In fact, $(f\otimes e_1 + g\otimes e_2) \in \mathcal{N}^\perp$ if and only if

$$g = c_1K(\cdot,w_2) \text{ and } f = c_2K(\cdot,w_1) + \bar{t}cK(\cdot,w_2),$$

for some scalars c_1 and c_2. Consequently,

$$\left\{\begin{pmatrix} K(\cdot, w_1) \\ 0 \end{pmatrix}, \begin{pmatrix} \bar{t}K(\cdot, w_2) \\ K(\cdot, w_2) \end{pmatrix}\right\}$$

forms a basis for $\mathcal{N}^\perp$. Clearly, the matrix representation of $(M_{z_l}^* \otimes I)_{|\mathcal{N}^\perp}$ with respect to this basis is of the form $\begin{pmatrix} \bar{w}_l^1 & 0 \\ 0 & \bar{w}_l^2 \end{pmatrix}$. Therefore, applying Lemma 5.15, we see that

$$[(M_{z_l}^* \otimes I)_{|\mathcal{N}^\perp}] = \begin{pmatrix} \bar{w}_l^1 & \frac{(\bar{w}_l^2 - \bar{w}_l^1)\bar{t}K_{1,2}}{\sqrt{K_{1,1}K_{2,2} + |t|^2(K_{1,1}K_{2,2} - |K_{1,2}|^2)}} \\ 0 & \bar{w}_l^2 \end{pmatrix},\ K_{i,j} := K(w_i, w_j),$$

w.r.t the orthonormal basis obtained applying the Gram-Schmidt process to the basis vectors found above. Consequently, we have

$$[P_{\mathcal{N}^\perp}\big(M_{z_l} \otimes I\big)_{|\mathcal{N}^\perp}] = \begin{pmatrix} w_l^1 & 0 \\ \frac{(w_l^2 - w_l^1)tK_{2,1}}{\sqrt{K_{1,1}K_{2,2} + |t|^2(K_{1,1}K_{2,2} - |K_{1,2}|^2)}} & w_l^2 \end{pmatrix}. \tag{5.4}$$

As before, we have proved the following.

Proposition 5.18. *The module tensor product $\mathcal{H}_K \otimes_{\mathcal{O}(\bar{\Omega})} C^2_{w_1,w_2}(t(w_2 - w_1))$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}(\tilde{t}(w_2 - w_1))$, where $\tilde{t}$ is the scalar given by*

$$\tilde{t} = \frac{tK_{2,1}}{\sqrt{K_{1,1}K_{2,2} + |t|^2(K_{1,1}K_{2,2} - |K_{1,2}|^2)}}.$$

Remark 5.19. Note that for any non zero t in $\mathbb{C}$, we have

$$|\tilde{t}| < \frac{|K_{2,1}|}{\sqrt{(K_{1,1}K_{2,2} - |K_{1,2}|^2)}} = \lambda_K \text{ (say)}.$$

It is easy to see that the set $\{|\tilde{t}| : t \in \mathbb{C}, t \neq 0\}$ is equal to the interval $(0, \lambda_K)$. As in the equal eigenvalue case discussed before, if we assume that the tuple of operator $\boldsymbol{M}$ acting on $\mathcal{H}_K$ is jointly subnormal with normal spectrum in Ŝilov boundary of $\overline{\Omega}$, then it follows from the Theorem of Sarason, Theorem 5.11, that the Hilbert module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$, where $|t| \in (0, \lambda_K)$, admits a boundary dilation.

5.2.1 Localization of operators in the Cowen-Douglas class

Let $\boldsymbol{T} = (T_1, \ldots, T_m)$ be a commuting m-tuple of operators in the class $B_1(\Omega^*)$. We therefore assume that $\boldsymbol{T}$ is jointly unitarily equivalent to the m-tuple of operators $\boldsymbol{M}^*$ defined on some Hilbert space $\mathcal{H}_K$ consisting of holomorphic function on Ω possessing a reproducing kernel

K (see [16]). We assume that $\mathscr{H}_K$ is a Hilbert module over the algebra $\mathscr{O}(\overline{\Omega})$ with the module action given by $r \cdot h = r(\boldsymbol{T})h$, $\|r \cdot h\| \le c\|r\|_\infty \|h\|$ for some $c > 0$.

Let w be an arbitrary but fixed point in Ω and $a = (a_1, a_2, \ldots, a_m)$ be a vector in $\mathbb{C}^m$. Consider the subspace $\mathscr{S}_w(a)$ of $\mathscr{H}_K$, defined by $\mathscr{S}_w(a) = \vee\{K(\cdot, w), \bar{\partial}_a K(\cdot, w)\}$, where $\bar{\partial}_a K(\cdot, w)$ denotes the vector $(\bar{a}_1 \bar{\partial}_1 K(\cdot, w) + \cdots + \bar{a}_m \bar{\partial}_m K(\cdot, w))$ in $\mathscr{H}_K$. Note that $\mathscr{S}_w(a)$ is a joint invariant subspace for the tuple $\boldsymbol{M}^*$. Since $(M_l^* - \bar{w})K(\cdot, w) = 0$ and $(M_l^* - \bar{w})(\bar{\partial}_a K(\cdot, w)) = \bar{a}_l K(\cdot, w)$, the matrix representation of $M_l^* \,|_{\mathscr{S}_w(a)}$ w.r.t this basis is of the form

$$[M^*_{l\,|\mathscr{S}_w(a)}] = \begin{pmatrix} \bar{w}_l & \bar{a}_l \\ 0 & \bar{w}_l \end{pmatrix}.$$

Applying Lemma 5.15, we see that the matrix representation of $P_{\mathscr{S}_w(a)} M_{l|\mathscr{S}_w(a)}$ is

$$[P_{\mathscr{S}_w(a)} M_{l|\mathscr{S}_w(a)}] = \begin{pmatrix} w_l & 0 \\ \frac{a_l}{\sqrt{\langle -\mathscr{K}_K(\bar{w})a, a\rangle}} & w_l \end{pmatrix},$$

w.r.t the orthonormal basis obtained from the Gram-schmidt orthonormalization of the two basis vectors identified above. Since $\mathscr{S}_w(a)$ is a co-invariant subspace of $\mathscr{H}_K$, the Hilbert module $\mathscr{S}_w(a)$ is a quotient module of $\mathscr{H}_K$. In fact from the matrix representation of the operator tuple $P_{\mathscr{S}_w(a)} \boldsymbol{M}_{|\mathscr{S}_w(a)}$, which induce the module action on $\mathscr{S}_w(a)$, it is easy to see that the module $\mathscr{S}_w(a)$ is isomorphic to the module $\mathbb{C}^2_w(\frac{a}{\sqrt{\langle -\mathscr{K}_K(\bar{w})a, a\rangle}})$.

In a similar fashion, for two fixed but arbitrary point w_1, w_2 in Ω, if we consider the co-invariant subspace $\mathscr{S}_{w_1,w_2} = \vee\{K(\cdot, w_1), K(\cdot, w_2)\}$, then it follows that the quotient module $\mathscr{S}_{w_1,w_2}$ of $\mathscr{H}_K$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1))$, where the scalar t is given by

$$t = \frac{K(w_2, w_1)}{\sqrt{K(w_1, w_1)K(w_2, w_2) - |K(w_1, w_2)|^2}}.$$

Remark 5.20. Let $a = (a_1, \ldots, a_m)$ be a vector in $\mathbb{C}^m$ and w_1 be an arbitrary point in Ω. Set $w_2 = w_1 + za$, where $z \in \mathbb{C}$ has been chosen in a way such that $w_2 \in \Omega$. The module $\mathscr{S}_{w_1,w_2}$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}(tza)$, where t is defined as above. Since the module $\mathbb{C}^2_{w_1,w_2}(tza)$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}(t|z|a)$, without loss of generality we can assume $w_2 = w_1 + |z|a$. The modules $\mathbb{C}^2_w(a)$ can be obtained from the modules $\mathbb{C}^2_{w_1,w_2}(t(w_2 - w_1)$ by a limiting process as shown below.

$$\lim_{z\to 0} t|z|a = \lim_{z\to 0} \frac{|z|K(w_2, w_1)a}{\sqrt{K(w_1, w_1)K(w_2, w_2) - |K(w_1, w_2)|^2}} = \frac{a}{\sqrt{\langle -\mathscr{K}(\bar{w}_1)a, a\rangle}}. \tag{5.5}$$

To verify this equality, it is sufficient to show

$$\lim_{z\to 0} \frac{K(w_1, w_1)K(w_2, w_2) - |K(w_1, w_2)|^2}{|z|^2} = (K(w_1, w_1))^2 \langle -\mathscr{K}(\bar{w}_1)a, a\rangle.$$

Note that

$$\Big\langle \frac{K(\cdot,w_2)-K(\cdot,w_1)}{\bar z}, \frac{K(\cdot,w_2)-K(\cdot,w_1)}{\bar z}\Big\rangle K(w_1,w_1) - \Big|\Big\langle K(\cdot,w_1), \frac{K(\cdot,w_2)-K(\cdot,w_1)}{\bar z}\Big\rangle\Big|^2$$
$$= \frac{\big(K(w_2,w_2)-K(w_1,w_2)-K(w_2,w_1)+K(w_1,w_1)\big)K(w_1,w_1)}{|z|^2} - \frac{|K(w_2,w_1)-K(w_1,w_1)|^2}{|z|^2}$$
$$= \frac{1}{|z|^2}\big(K(w_2,w_2)K(w_1,w_1)-|K(w_1,w_2)|^2\big).$$

Since $\lim_{z\to 0}\frac{K(\cdot,w_2)-K(\cdot,w_1)}{\bar z} = \lim_{z\to 0}\frac{K(\cdot,(w_1+za))-K(\cdot,w_1)}{\bar z} = \bar\partial_a K(\cdot,w_1)$, taking limit on the both side, as $z\to 0$ in the equation displayed above, we see that

$$\lim_{z\to 0}\frac{K(w_1,w_1)K(w_2,w_2)-|K(w_1,w_2)|^2}{|z|^2}$$
$$= \|\bar\partial_a K(\cdot,w_1)\|^2\|K(\cdot,w_1)\|^2 - |\langle \bar\partial_a K(\cdot,w_1), K(\cdot,w_1)\rangle|^2$$
$$= (K(w_1,w_1))^2\langle -\mathscr{K}(\bar w_1)a,a\rangle.$$

Assume that the m-tuple $\boldsymbol{M}$ is jointly subnormal with joint spectrum $\sigma(\boldsymbol{M})$ contained in $\overline{\Omega}$ with normal spectrum contained in Ŝilov boundary. By Theorem 5.11 due to Sarason, the quotient modules $\mathscr{S}_w(a)$ and the module tensor product $\mathscr{H}_K\otimes_{\mathscr{O}(\overline{\Omega})}\mathbb{C}^2_w(a)$ admit a boundary dilation. The quotient module $\mathscr{S}_w(a)$ is isomorphic to the module $\mathbb{C}^2_w(\frac{a}{\sqrt{\langle -\mathscr{K}_K(\bar w)a,a\rangle}})$ and the module $\mathscr{H}_K\otimes_{\mathscr{O}(\overline{\Omega})}\mathbb{C}^2_w(a)$ is isomorphic to $\mathbb{C}^2_w(\frac{a}{\sqrt{1-\langle \mathscr{K}_K(\bar w)a,a\rangle}})$. Consequently, these modules admit a boundary dilation.

Also, for a vector $v\in\mathbb{C}^m$, the module $\mathbb{C}^2_w(v)$ is contractive if and only if $C_{\Omega,w}(v)\le 1$ as shown in Lemma 5.5. Since these contractive modules are completely contractive (cf. [4]), they must possess a $\partial\mathscr{O}(\overline{\Omega})$ dilation guaranteed by a theorem due to Arveson [5]. It is therefore natural to ask if there exists a Hilbert module $\mathscr{H}_K$ that admits $\partial\mathscr{O}(\overline{\Omega})$ dilation so that the contractive module $\mathbb{C}^2_w(v)$ is isomorphic to $\mathbb{C}^2_w(\frac{a}{\sqrt{\langle -\mathscr{K}_K(\bar w)a,a\rangle}})$ or $\mathbb{C}^2_w(\frac{a}{\sqrt{1-\langle \mathscr{K}_K(\bar w)a,a\rangle}})$, for some $a\in\mathbb{C}^m$. For a fixed point w in the Euclidean ball $\mathbb{B}^m$, we find such a Hilbert module as long as $C_{\Omega,w}(v)<1$. However, when $C_{\Omega,w}(v)=1$, the answer is not obvious.

Without loss of generality, we work with the point $w=0$ in $\mathbb{B}^m$ since it can be moved to any other point w using a biholomorphic automorphism of the ball. The module $\mathbb{C}^2_w(v)$ is contractive if and only if $C_{\Omega,w}(v)=\|v\|_2\le 1$. Consider the case when $\|v\|_2<1$, then we will construct explicit boundary dilation for the module $\mathbb{C}^2_w(v)$.

The case of $m=1$ is included in the discussion to follow in the next section. Therefore, in the discussion below, we assume $m>1$. First, let us consider the case of $v=\lambda e_1$, $|\lambda|<1$, where $e_1 = (1,0,\ldots,0) \in \mathbb{C}^m$. Let $\mathscr{H}_S$ denote the Hardy space on the Euclidean ball $\mathbb{B}^m$. The reproducing kernel of the Hardy space, namely, the Sžego kernel $S(z,w)$ is of the form

$$(1-\langle z,w\rangle)^{-m} = \sum_I a_I z^I \bar{w}^I,\ a_I = \frac{m(m+1)\ldots(m+|I|-1)}{I!}.$$

As is well known, the Hardy space admits a realization as a closed subspace in the $L^2(\partial\Omega, d\mu)$, where μ is the normalized surface area measure on $\partial\Omega$. The operator tuple $\boldsymbol{M}$ is then jointly subnormal. Thus $\|f\|^2_{\mathscr{H}_S} = \int_{\partial\Omega}|f|^2 d\mu$, for every $f \in \mathscr{H}_S$. Clearly, the set of monomials $\{z^I : I \in \mathbb{Z}_+^m\}$ is an orthogonal basis in $\mathscr{H}_S$ and $\|z^I\|^2_{d\mu} = {a_I}^{-1}$. Fix a natural number k in $\mathbb{N}$ and consider the measure $d\mu_k(z) = |z_1|^{2k} d\mu(z)$ on $\partial\Omega$. Note that

$$\langle z^I, z^J\rangle_{d\mu_k} = \langle z^{I+k\epsilon_1}, z^{J+k\epsilon_1}\rangle_{d\mu}.$$

Let $P^2(\mu_k)$ denote the closure of all polynomials in $L^2(\mu_k)$. Note that in $P^2(\mu_k)$, the set of monomials $\{z^I : I \in \mathbb{Z}_+^m\}$ is an orthogonal basis and

$$\|z^I\|^2_{d\mu_k} = \frac{(I+k\epsilon_1)!}{m(m+1)\ldots(m+|I|+k-1)}.$$

The tuple of operators $\boldsymbol{M}$ is clearly jointly subnormal. Note that μ_k is a Reinhardt measure on $\partial\mathbb{B}^m$ with support equal to $\partial\mathbb{B}^m$. It follows from a result of Curto (cf. [17, Theorem 3]) on spectral theory of Reinhardt measure that $\boldsymbol{M}^*$ on $P^2(\mu_k)$ belongs to $B_1(\Omega^*)$. Let K denote the kernel function of the Hilbert space $P^2(\mu_k)$. A straightforward computation shows that

$$\langle \mathscr{K}_K(0)e_1, e_1\rangle = -\frac{m+k}{k+1}.$$

Since $\frac{m+k}{k+1} \to 1$ as $k \to \infty$ and $|\lambda| < 1$, there exists a $k \in \mathbb{N}$ such that $\frac{\sqrt{m+k}}{\sqrt{k+1}} < \frac{1}{|\lambda|}$. Depending on this $k \in \mathbb{N}$, consider the associated Hilbert module $\mathscr{H}(\mu_k) = P^2(\mu_k)$ as above. Notice that $|\lambda| < \frac{1}{\sqrt{-\langle \mathscr{K}_K(0)e_1, e_1\rangle}}$. Following the remark 5.17, it is easy to see that the module $\mathbb{C}^2_w(\lambda e_1)$ is isomorphic to the module tensor product $\mathscr{H}(\mu_k) \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_w(ce_1)$ for an appropriate choice of $c \in \mathbb{C}$. The module $\mathbb{C}^2_w(\lambda e_1)$ is the compression to a co-invariant subspace of $\mathscr{H}(\mu_k) \otimes \mathbb{C}^2$ by definition. Thus $\mathbb{C}^2_w(\lambda e_1)$ admits a boundary dilation, namely, $L^2(\partial\mathbb{B}^m, \mu_k) \otimes \mathbb{C}^2$.

For an arbitrary point $w \in \mathbb{B}^m$ and $v \in \mathbb{C}^m$, let b denote the unit vector of carathéodory norm that is $b = \frac{v}{C_{\mathbb{B}^m,w}(v)}$. Let φ be a bi-holomorphic automorphism of $\mathbb{B}^m$ such that $\varphi(w) = 0$ and $\overline{D\varphi(w)}b = e_1$. Now consider the kernel function $\tilde{K}(z,w) = K(\varphi(z), \varphi(w))$ on $\mathbb{B}^m$ where $K(z,w)$ is the kernel function for which $\langle \mathscr{K}_K(0)e_1, e_1\rangle = -\frac{m+k}{k+1}$. Note that $\mathscr{H}_{\tilde{K}} = \{f \circ \varphi : f \in \mathscr{H}_K\}$ and the map $f \mapsto f \circ \varphi$ is an isometry from $\mathscr{H}_K$ onto $\mathscr{H}_{\tilde{K}}$. So, we have

$$\begin{aligned}\|f\circ\varphi\|^2_{\mathscr{H}_{\tilde{K}}} &= \int_{\partial\mathbb{B}^m} |f(z)|^2 d\mu_k(z)\\ &= \int_{\partial\mathbb{B}^m} |f\circ\varphi(z)|^2 d(\mu_k \circ \varphi(z)).\end{aligned}$$

Hence the tuple of operator $\boldsymbol{M}$ on $\mathscr{H}_{\tilde{K}}$ is again jointly subnormal and $\boldsymbol{M}^*$ belongs to $B_1(\Omega^*)$. In fact the tuple of operator $\boldsymbol{M}$ on $\mathscr{H}_{\tilde{K}}$ is unitarily equivalent to $\varphi^{-1}(\boldsymbol{M})$ on $\mathscr{H}_K$. A straightforward computation gives

$$\mathscr{K}_{\tilde{K}}(w) = (D\varphi(w))^{\mathrm{tr}}\mathscr{K}_K(0)\overline{D\varphi(w)}.$$

Consequently, we have $\langle \mathscr{K}_{\tilde{K}}(w)b, b\rangle = \langle \mathscr{K}_K(0)e_1, e_1\rangle = -\frac{m+k}{k+1}$. Now repeating the same argument as before, we conclude that the module $\mathbb{C}^2_w(v)$ must possess a boundary dilation, namely, $L^2(\partial\mathbb{B}^m, \mu_k \circ \varphi) \otimes \mathbb{C}^2$. We have therefore proved the following theorem.

Theorem 5.21. *Let $w \in \mathbb{B}^m$ be a fixed but arbitrary point. If $v \in \mathbb{C}^m$ is a vector with $C_{\mathbb{B}^m,w}(v) < 1$, then the module $\mathbb{C}^2_w(v)$ is contractive over $\mathscr{O}(\bar{\mathbb{B}}^m)$ and admits a $\partial\mathscr{O}(\bar{\mathbb{B}}^m)$ dilation to the module $L^2(\partial\mathbb{B}^m, \mu_k \circ \varphi_w) \otimes \mathbb{C}^2$.*

Now consider the module $\mathbb{C}^2_w(v)$ for some v in $\mathbb{C}^m$ with $C_{\mathbb{B}^m,0}(v) = \|v\|_2 = 1$. It follows from the remark 5.17 that the module tensor product obtained above does not produce a boundary dilation. We ask if the module $\mathbb{C}^2_w(v)$, $\|v\| = 1$, is isomorphic to the $\mathbb{C}^2_w(\frac{v}{\sqrt{-\langle \mathscr{K}_K(0)v, v\rangle}})$ for an appropriate choice of Hilbert module $\mathscr{H}_K$, which is jointly subnormal. Given a vector v, $\|v\|_2 = 1$, the question is to find a (jointly subnormal) Hilbert module $\mathscr{H}_K$ with the property: $\langle \mathscr{K}_K(0)v, v\rangle = -1$.

Indeed, if Ω is a bounded planar domain and $w \in \Omega$, then a version of this question amounts to the existence of the extremal operator at w, see [26]. However, in the case of the Euclidean Ball $\mathbb{B}^m$, $m > 1$, we did not succeed in finding a commuting m- tuple of jointly subnormal operators $\boldsymbol{T}$ in $B_1(\Omega^*)$ such that $\langle \mathscr{K}_K(0)v, v\rangle = -1$, $\|v\|_2 = 1$. However, if we drop the requirement of "joint subnoramlity", then the adjoint of the multiplication by the co-ordinate functions on the Drury-Arveson space $(\mathscr{H}, (1 - \langle z, w\rangle)^{-1})$ is an extremal operator since its curvature at 0 is I_m.

5.2.2 Extremal problem and curvature inequality

Let $\boldsymbol{M}$ be a commuting m-tuple of jointly subnormal operator acting on a reproducing kernel Hilbert space $\mathscr{H}_K$. Let the joint spectrum $\sigma(\boldsymbol{M})$ of $\boldsymbol{M}$ be equal to $\overline{\Omega}$ and the normal spectrum $\sigma_\perp(\boldsymbol{M})$ be a subset of $\overline{\Omega}$. Assume that the m-tuple of operators $\boldsymbol{M}^* \in B_1(\Omega^*)$. Let μ be the scalar spectral measure, supported on $\overline{\Omega}$, of the minimal normal extension of $\boldsymbol{M}$. Thus $\|f\|^2 = \int |f(z)|^2 d\mu(z)$, for all $f \in \mathscr{H}_K$.

Let $\boldsymbol{M^*}$ be an m-tuple of operators in $B_1(\Omega^*)$ acting on some reproducing kernel Hilbert space $\mathscr{H}_k$. Assume that the Hilbert module $\mathscr{H}_K$ over the algebra $\mathscr{O}(\overline{\Omega})$ is contractive. Then for any fixed but arbitray point $w \in \Omega$ and any vector $v \in \mathbb{C}^m$, we have the curvature inequality

$$\langle \mathscr{K}_{M^*}(\bar{w})v, v\rangle \le -C_{\Omega,w}(v)^2.$$

For a fixed $v \in \mathbb{C}^m$, we say that the Hilbert module $\mathscr{H}_K$ is extremal at $\bar{w} \in \Omega^*$, if $\langle \mathscr{K}_{M^*}(\bar{w})v, v\rangle = -C_{\Omega,w}(v)^2$. For the case of planar domains, the existence of such extremal operators is established in [26]. For the multivariate case, the question of the existence of such extremal operators is not completely resolved. We find a tractable necessary condition for a jointly subnormal Hilbert module $\mathscr{H}_K$ to be extremal assuming that the kernel function K_w extends, as a holomorphic function, to a neighbourhood of $\overline{\Omega}$. We haven't been bale to find a Hilbert module with these properties in the multivariate case. None the less, we see many naturally occurring examples of Jointly subnormal Hilbert modules can't be extremal at $\bar{w}$, as long as K_w is assumed to extend to a neigbourhood of $\overline{\Omega}$.

Let w be an arbitrary but fixed point in Ω, and v be a vector in $\mathbb{C}^m$. Let $\mathscr{M}_w(v)$ be the closed convex set in $\mathscr{H}_K$ defined by $\mathscr{M}_w(v) = \{f \in \mathscr{H}_K : f(w) = 0, (\nabla f(w).v) = 1\}$, where $(\nabla f(w).v) = v_1\frac{\partial f}{\partial z_1}(w) + \ldots + v_m\frac{\partial f}{\partial z_m}(w)$. Consider the following extremal problem

$$\inf\{\|f\|^2 : f \in \mathscr{M}_w(v)\}.$$

Let $\mathscr{E}_w(v)$ be the subspace of $\mathscr{H}_K$ defined by

$$\mathscr{E}_w(v) = \{f \in \mathscr{H}_K : f(w) = 0, (\nabla f(w).v) = 0\}.$$

Since $f+g \in \mathscr{M}_w(v)$, whenever $f \in \mathscr{M}_w(v)$ and $g \in \mathscr{E}_w(v)$, It is evident that the unique function F which solves the extremal problem must belong to $\mathscr{E}_w(v)^\perp$. From the reproducing property of K, it follows that

$$f(w) = \langle f, K(\cdot, w)\rangle,\ (\nabla f(w).v) = \langle f, \bar{v}_1\bar{\partial}_{z_1}K(\cdot, w) + \ldots + \bar{v}_m\bar{\partial}_{z_m}K(\cdot, w)\rangle.$$

Consequently, we have $\mathscr{E}_w(v)^\perp = \vee\{K(\cdot, w), \bar{v}_1\bar{\partial}_{z_1}K(\cdot, w) + \cdots + \bar{v}_m\bar{\partial}_{z_m}K(\cdot, w)\}$. Let γ_1, γ_2 denote the vectors $K(\cdot, w), \bar{v}_1\bar{\partial}_{z_1}K(\cdot, w) + \cdots + \bar{v}_m\bar{\partial}_{z_m}K(\cdot, w)$ respectively. Let $F = c_1\gamma_1 + c_2\gamma_2$ be the solution of the extremal problem. Since $F \in \mathscr{M}_w(v)$, we have

$$\begin{aligned} c_1\gamma_1 + c_2\gamma_2(w) = 0 \text{ and } (\nabla c_1\gamma_1 + c_2\gamma_2(w).v) = 1, \\ \langle c_1\gamma_1 + c_2\gamma_2, \gamma_1\rangle = 0 \text{ and } \langle c_1\gamma_1 + c_2\gamma_2, \gamma_2\rangle = 1. \end{aligned}$$

Let G denotes the grammian matrix $(\!(\langle\gamma_j, \gamma_i\rangle)\!)_{i,j=1}^2$ and c denotes the column vector $(c_1, c_2)^{tr}$. we have $Gc = (0,1)^{tr} = e_2$. Thus $c = G^{-1}e_2$. Consequently, we have that

$$\begin{aligned} \|F\|^2 &= \|c_1\gamma_1 + c_2\gamma_2\|^2 \\ &= \langle Gc, c\rangle \\ &= \langle G^{-1}e_2, e_2\rangle \\ &= \frac{\|\gamma_1\|^2}{\|\gamma_1\|^2\|\gamma_2\|^2 - |\langle\gamma_1, \gamma_2\rangle|^2} \\ &= (K(w,w)\langle -\mathscr{K}_K(\bar{w})v, v\rangle)^{-1}. \end{aligned}$$

Hence we have

$$\inf\{\|f\|^2 : f \in \mathscr{M}_w(v)\} = (K(w,w)\langle -\mathscr{K}_K(\bar{w})v, v\rangle)^{-1}.$$

Recall that the Carathéodory norm $C_{\Omega,w}(v)$ of a vector v in $\mathbb{C}^m$ is obtained by solving the extremal problem

$$\sup\{|(\nabla f \cdot v)| : f \in \mathscr{A}(\Omega), \|f\|_\infty \le 1, f(w) = 0\}.$$

The existence of a holomorphic function $F_{w,v}(z)$ on Ω such that $F_{w,v}(w) = 0$, $\|F_{w,v}\|_\infty \le 1$ and $|(\nabla F_{w,v}.v)| = C_{\Omega,w}(v)$ follows from the Montel's theorem. Consider the function g defined by

$$g(z) := \frac{F_{w,v}(z)K(z,w)}{K(w,w)C_{\Omega,w}(v)}.$$

Since g is in $\mathscr{M}_w(v)$, we have the inequality

$$\begin{aligned}
\frac{1}{K(w,w)\langle -\mathscr{K}_K(\bar{w})v, v\rangle} &\le \|g\|^2 \\
&= \frac{1}{K(w,w)^2(C_{\Omega,w}(v))^2}\int_{\partial\Omega}|F_{w,v}(z)|^2|K(z,w)|^2 d\mu(z) \\
&\le \frac{1}{K(w,w)^2(C_{\Omega,w}(v))^2}\int_{\partial\Omega}|K(z,w)|^2 d\mu(z), \\
&= \frac{1}{K(w,w)(C_{\Omega,w}(v))^2},
\end{aligned}$$

where the last but one inequality follows from the inequality $\|F_{w,v}\|_\infty \le 1$. Hence we have $(C_{\Omega,w}(v))^2 \le \langle -\mathscr{K}_K(\bar{w})v, v\rangle$, which is the desired curvature inequality. So, in case of equality, g must solve the extremal problem and μ must satisfy $\mu(X^c \cap \overline{\Omega}) = 0$, where X is given by

$$X = \{z \in \overline{\Omega} : K_w(z) = 0\} \cup \{z \in \overline{\Omega} : |F_{w,v}(z)| = 1\}.$$

This provides an algorithm to construct examples of contractive modules for which the curvature inequality is strict.

5.3 Bundle shifts of rank 1 and localization: Planar domain

Let Ω be a finitely connected bounded planar domain whose boundary consists of $n+1$ analytic jordan curves and α be a character of the fundamental group $\pi_1(\Omega)$. Abrahamse and Douglas in [2] have shown that the adjoint of any multiplicity one bundle shift of index α over a planar domain Ω is in the Cowen-Douglas class $B_1(\Omega^*)$. Consequently, any such bundle shift can be realized as a multiplication, by the coordinate function, operator M on the

Hilbert space $\mathscr{H}_{K^\alpha}$, where $K^{(\alpha)}$ is a non-negative definite kernel defined on $\Omega\times\Omega$ determined by the character α. The bundle shifts M are pure subnormal operators of multiplicity one, the spectrum $\sigma(M)=\overline{\Omega}$ and the normal spectrum $\sigma_\perp(M)=\partial\Omega$. (cf. [2]). The operator M, being subnormal, admits $\overline{\Omega}$ as a spectral set. Consequently $\mathscr{H}_{K^\alpha}$ is a contractive Hilbert module over the function algebra $\mathscr{O}(\overline{\Omega})$ induced by the multiplication operator M.

In what follows describe the localization of the operator M^* in the Cowen-Douglas class $B_1(\Omega^*)$. This could be easily obtained from the multi-variable case discussed in the previous section. However, it is repeated here for clarity. Let w be a fixed but arbitrary point in Ω. Let $\mathscr{S}_w^\alpha\subseteq\mathscr{H}_{K^\alpha}$ be the co-invariant subspace of M defined by

$$\mathscr{S}_w^\alpha := \{f\in\mathscr{H}_{K^\alpha} \mid f(w)=0, f'(w)=0\} = \big[\vee\{K_w^\alpha,\bar\partial K_w^\alpha\}\big]^\perp. \tag{5.6}$$

The module action for the quotient module $\mathscr{S}_w^\alpha$ is induced by the operator $(P_{\mathscr{S}_w^\alpha}M)_{|\mathscr{S}_w^\alpha}$ and the matrix of the operator $(P_{\mathscr{S}_w^\alpha}M)_{|\mathscr{S}_w^\alpha}$ w.r.t an orthonormal basis is of the form

$$[(P_{\mathscr{S}_w^\alpha}M)_{|\mathscr{S}_w^\alpha}] = \begin{pmatrix} w & 0 \\ \frac{1}{\sqrt{-\mathscr{K}_{K^\alpha}(\bar w)}} & w \end{pmatrix}. \tag{5.7}$$

So, the Hilbert module $\mathscr{S}_w^\alpha$ is isomorphic to the module $\mathbb{C}_w^2(\frac{1}{\sqrt{-\mathscr{K}_{K^\alpha}(\bar w)}})$. Since the Hilbert module $\mathscr{H}_{K^\alpha}$ over $\mathscr{O}(\overline{\Omega})$ induced by the operator M is contractive, the quotient module $\mathscr{S}_w^\alpha$ is also contractive. Using lemma 5.5 we have $C_{\Omega,w}(\frac{1}{\sqrt{-\mathscr{K}_{K^\alpha}(\bar w)}})\le 1$. In the case of a palnar domain Ω, Carathéodory norm, $C_{\Omega,w}(\lambda)$ is related to the Sz̈ego kernel $S_\Omega(w,w)$ for the domain Ω in the following way (cf. [7, Theorem 13.1])

$$C_{\Omega,w}(\lambda) = |\lambda|\sup\{|f'(w)| : f\in\mathscr{O}(\overline{\Omega}), f(w)=0, \|f\|_\infty\le 1\} = |\lambda| 2\pi S_\Omega(w,w).$$

Consequently, we obtain the following inequality for the curvature of the adjoint M^*

$$\mathscr{K}_{K^\alpha}(\bar\xi)\le -4\pi^2(S_\Omega(\xi,\xi))^2,\ \ \xi\in\Omega.$$

It is useful to restate the curvature inequality in the equivalent form:

$$\frac{\partial^2}{\partial z\partial\bar z}\log K^\alpha(z,z)|_{z=\xi}\ge 4\pi^2(S_\Omega(\xi,\xi))^2,\ \ \xi\in\Omega.$$

For a fixed point w in Ω, Misra (See [26]) has shown the existence of a bundle shift of index β (depending upon w) such that $\frac{\partial^2}{\partial z\partial\bar z}\log K^\beta(z,z)|_{z=w} = 4\pi^2(S_\Omega(w,w))^2$. Consequently it follows that

$$\inf_\alpha \frac{\partial^2}{\partial z\partial\bar z}\log K^\alpha(z,z)|_{z=w} = 4\pi^2(S_\Omega(w,w))^2 = -\sup_\alpha \mathscr{K}_{K^\alpha}(\bar w).$$

Let's consider the interval $\mathscr{I}_\Omega(w) \subseteq \mathbb{R}_+$ defined by

$$\mathscr{I}_\Omega(w) := [\frac{1}{\sqrt{J_\Omega(w)}}, \frac{1}{2\pi S_\Omega(w,w)}], \tag{5.8}$$

where $J_\Omega(w)$ is given by $J_\Omega(w) = \sup_\alpha \frac{\partial^2}{\partial z \partial \bar{z}} \log K^\alpha(z,z)|_{z=w} = -\inf_\alpha \mathscr{K}_{K^\alpha}(\bar{w})$. By definition, $(-\mathscr{K}_{K^\alpha}(\bar{w}))^{-\frac{1}{2}} \in \mathscr{I}_\Omega(w)$. For a fixed w in Ω, consider the function $\phi : \mathbb{T}^n \mapsto \mathbb{R}$ defined by $\phi(\alpha) = \mathscr{K}_{K^\alpha}(\bar{w})$. We show that ϕ is a continuous function. Consequently, the set $\{(-\mathscr{K}_{K^\alpha}(\bar{w}))^{-\frac{1}{2}} : \alpha \in \mathbb{T}^n\}$ is equal to the interval $\mathscr{I}_\Omega(w)$. (Recall that the characters α are in one to one correspondence with elements of the torus $\mathbb{T}^n$ equivalently also in one to one correspondence with elements of $[0,1)^n$.)

Lemma 5.22. *For a fixed w in Ω, the function $\phi : \mathbb{T}^n \to \mathbb{R}$ defined by $\phi(\alpha) = \mathscr{K}_{K^\alpha}(\bar{w})$ is continuous.*

Proof. We have that

$$\mathscr{K}_{K^\alpha}(\bar{w}) = -\frac{\partial^2}{\partial z \partial \bar{z}} \log K^\alpha(z,z)|_{z=w} = -\frac{\|K_w^\alpha\|^2 \|\bar{\partial} K_w^\alpha\|^2 - |\langle K_w^\alpha, \bar{\partial} K_w^\alpha \rangle|^2}{(K^\alpha(w,w))^2}, \;\; w \in \Omega.$$

For a fixed w in Ω, Widom in [44], has shown that the function $\alpha \to K_w^\alpha$ is continuous. Hence to prove that ϕ is continuous, it is sufficient to show that for each fixed w in Ω the function $\alpha \to \bar{\partial} K_w^\alpha$ is continuous. The technique of the proof below is not very different from the one given in [20, Ch.5, Proposition 4.7] to show that the function $\alpha \to K_w^\alpha$ is continuous. Thus to complete the proof of the continuity of the map $\alpha \to K_w^\alpha$, we recall the proof from [20, Ch.5, Proposition 4.7] and modify it slightly wherever necessary.

Let $f_n(z) = \bar{\partial} K_w^{\alpha_n}(z)$ for $z \in \overline{\Omega}$ and assume $\alpha_n \to \alpha$ in $\mathbb{T}^n$. We show that $\{f_n\}$ is uniformly norm bounded in $L^2(\partial\Omega, d\omega)$ and consequently, $\{f_n\}$ is uniformly bounded on compact subset of Ω. If E is a compact set in Ω, then there is a constant C depending only on E and Ω such that

$$\sup_{\xi \in E} |r(\xi)| \leq C \|r\|, \;\; r \in H_\alpha^2(\Omega). \tag{5.9}$$

Now using Cauchy integral formula, we find a constant C_1 depending only on E and Ω such that

$$\sup_{\xi \in E} |r'(\xi)| \leq C_1 \|r\|, \;\; r \in H_\alpha^2(\Omega), \;\; \alpha \in \mathbb{T}^n. \tag{5.10}$$

So for the compact set E containing w in Ω, we have

$$\|f_n\| = \sup_{\|r\| \leq 1} |\langle f_n, r \rangle| = \sup_{\|r\| \leq 1} |r'(w)| \leq C_1.$$

Therefore, we may assume without loss of generality that the sequence of function $\{f_n\}$ converges weakly in $L^2(\partial\Omega, d\omega)$ to some function h. It also follows from (5.10) that $\{f_n\}$ is uniformly bounded on compact sets in Ω. Consequently we may assume $f_n \to f$ uniformly on compact sets in Ω for some f in $H^2_\alpha(\Omega)$. There exist of a $\delta > 0$ and a sequence g_n of multiplicative functions satisfying

$$g_n \in H^\infty_{\alpha-\alpha_n}(\Omega),\ \ \delta \le |g_n(z)| \le \frac{1}{\delta},\ \ z \in \overline{\Omega},$$

with the property that $g_n \to 1$ not only pointwise in Ω but also in $L^2(\partial\Omega, d\omega)$, see [20, Ch.5, Proposition 4.7]. It also follows from (5.10) that $(g_n)' \to 0$ pointwise. Thus for $z \in \Omega$, we have

$$\begin{aligned}\int_{\partial\Omega} h\overline{K}^\alpha_z d\omega &= \lim \int_{\partial\Omega} f_n \overline{K}^\alpha_z d\omega \\ &= \lim \int_{\partial\Omega} f_n g_n \overline{K}^\alpha_z d\omega \\ &= \lim f_n(z) g_n(z) = f(z).\end{aligned}$$

A similar argument shows that h lies in $H^2_\alpha(\Omega)$ and thus $f = h$ on $\partial\Omega$. Now we show that f is equal to $\bar{\partial}K^\alpha_w$. Let $g \in H^\infty_\alpha(\Omega)$ and note that

$$\begin{aligned}\int_{\partial\Omega} g\bar{f} d\omega &= \lim \int_{\partial\Omega} g\bar{f}_n d\omega \\ &= \lim \int_{\partial\Omega} g\frac{1}{g_n}\bar{f}_n d\omega \\ &= \lim (\frac{g}{g_n})'(w) \\ &= g'(w).\end{aligned}$$

Since $H^\infty_\alpha(\Omega)$ is dense in $H^2_\alpha(\Omega)$ we get that $f = \bar{\partial}K^\alpha_w$. □

Let w_1, w_2 be two fixed but arbitrary point in Ω. Consider the subspace $\mathscr{S}^\alpha_{w_1,w_2}$ of $\mathscr{H}_{K^\alpha}$ defined by

$$\mathscr{S}^\alpha_{w_1,w_2} := \{f \in \mathscr{H}_{K^\alpha} \mid f(w_1) = 0, f(w_2) = 0\} = \big[\vee \{K^\alpha_{w_1}, \bar{\partial}K^\alpha_{w_2}\}\big]^\perp. \tag{5.11}$$

Module action of the quotient module $\mathscr{S}^\alpha_{w_1,w_2}$ is induced by the operator $(P_{\mathscr{S}^\alpha_{w_1,w_2}} M)_{|\mathscr{S}^\alpha_{w_1,w_2}}$ and the matrix of the operator $(P_{\mathscr{S}^\alpha_{w_1,w_2}} M)_{|\mathscr{S}^\alpha_{w_1,w_2}}$ w.r.t an orthonormal basis is of the form:

$$[(P_{\mathscr{S}^\alpha_{w_1,w_2}} M)_{|\mathscr{S}^\alpha_{w_1,w_2}}] = \begin{pmatrix} w_1 & 0 \\ (w_2 - w_1)\lambda_{w_1,w_2}(\alpha) & w_2 \end{pmatrix}, \tag{5.12}$$

where $\lambda_{w_1,w_2}(\alpha) = \frac{K^\alpha(w_2,w_1)}{\sqrt{K^\alpha(w_1,w_1)K^\alpha(w_2,w_2)-|K^\alpha(w_1,w_2)|^2}}$. So, the quotient module $\mathscr{S}^\alpha_{w_1,w_2}$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda_{w_1,w_2}(\alpha))$. Since the Hilbert module $\mathscr{H}_{K^\alpha}$ over $\mathscr{O}(\bar{\Omega})$ induced by the operator M is contractive, the quotient module $\mathscr{S}^\alpha_{w_1,w_2}$ is also contractive. Using lemma 5.7 we obtain $|\lambda_{w_1,w_2}(\alpha)|^2 \le \frac{1-(m_\Omega(w_1,w_2))^2}{(m_\Omega(w_1,w_2))^2}$. Consequently, we have

$$\sup_{\alpha\in\mathbb{T}^n} |\lambda_{w_1,w_2}(\alpha)| \le \frac{\sqrt{1-(m_\Omega(w_1,w_2))^2}}{m_\Omega(w_1,w_2)} = t_\Omega(w_1,w_2) \text{ (say)}.$$

It has been shown that there exist a multiplicty one bundle shift of index α for which we have $|\lambda_{w_1,w_2}(\alpha)| = t_\Omega(w_1,w_2)$ (cf. [8, Pg. 1188]). Let $p_\Omega(w_1,w_2) := \inf_\alpha |\lambda_{w_1,w_2}(\alpha)|$ and consider the interval

$$\mathscr{I}_\Omega(w_1,w_2) := \big[|w_1-w_2|p_\Omega(w_1,w_2), |w_1-w_2|t_\Omega(w_1,w_2)\big] \subseteq \mathbb{R}_+. \tag{5.13}$$

By definition $|(w_1-w_2)\lambda_{w_1,w_2}(\alpha)| \in \mathscr{I}_\Omega(w_1,w_2)$. For a fixed $w\in\Omega$, the function $\psi : \alpha \to K^\alpha_w$ is continuous. It therefore follows that the function $\lambda_{w_1,w_2} : \alpha \to \lambda_{w_1,w_2}(\alpha)$ is a continuous real valued function on $\mathbb{T}^n$. Hence the set $\{\lambda_{w_1,w_2}(\alpha) \mid \alpha\in\mathbb{T}^n\}$ is equal to the interval $\mathscr{I}_\Omega(w_1,w_2)$.

5.4 Constructing dilation for planar algebras

Let Ω be a finitely connected bounded domain in the complex plane $\mathbb{C}$ whose boundary consists of $n+1$ analytic Jordan curves. Let w, w_1, w_2 are fixed but arbitrary points in Ω and λ be a non zero number in $\mathbb{C}$. Consider the Hilbert modules $\mathbb{C}^2_w(\lambda)$ and $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ over $\mathscr{O}(\bar{\Omega})$, as discussed in example 5.4 and 5.6. Assume that the modules $\mathbb{C}^2_w(\lambda)$ and $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ are contarctive. Agler has shown that such contractive modules are completely contractive (cf. [4]). Consequently, following Arveson's deep result (cf. [5]), this modules admits a $\partial\Omega$ dilation. An explicit dilation for such modules is given in [8], see also [26]. We discuss below some finer details of this construction.

Equal eigenvalue case: First we consider the case of $\mathbb{C}^2_w(\lambda)$. Assume that the Hilbert module $\mathbb{C}^2_w(\lambda)$ is contractive over $\mathscr{O}(\bar{\Omega})$. Using Lemma 5.5 we have $C_{\Omega,w}(\lambda) \le 1$. In the case of a planar domain Ω, Carathéodory norm $C_{\Omega,w}(\lambda)$, as discussed in previous section, is related to the Szëgo kernel $S_\Omega(w,w)$ for the domain Ω in the following way (cf. [7, Theorem 13.1])

$$C_{\Omega,w}(\lambda) = |\lambda|\sup\{|f'(w)| : f\in\mathscr{O}(\bar{\Omega}), f(w)=0, \|f\|_\infty \le 1\} = |\lambda| 2\pi S_\Omega(w,w).$$

Consequently, we have $|\lambda| \le \frac{1}{2\pi S_\Omega(w,w)}$. Now two cases arise, which we describe below.

Case 1: $|\lambda| \in \mathscr{I}_\Omega(w)$, see (5.8) for the definition of the interval $\mathscr{I}_\Omega(w)$. From Lemma (5.22), the existence of a character α such that the module $\mathbb{C}^2_w(\lambda)$ is isomorphic to the module $\mathbb{C}^2_w(\frac{1}{\sqrt{(-\mathscr{K}_{K^\alpha}(\bar{w}))}})$ follows. Equivalently, the module $\mathbb{C}^2_w(\lambda)$ is isomorphic to the quotient module $\mathscr{S}^\alpha_w$ of the Hilbert module $\mathscr{H}_{K^\alpha}$ defined in (5.6).

Case 2: $0 < |\lambda| \le \frac{1}{2\pi S_\Omega(w,w)}$ but $|\lambda| \notin \mathscr{I}_\Omega(w)$. Now consider the module tensor product $\mathscr{H}_{K^\alpha} \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_w(a)$. Following (5.3), we find that the module $\mathscr{H}_{K^\alpha} \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_w(a)$ is isomorphic to the module $\mathbb{C}^2_w(\frac{a}{\sqrt{1-|a|^2\mathscr{K}_{K^\alpha}(\bar{w})}})$. Notice that

$$0 \le \frac{|a|}{\sqrt{1-|a|^2\mathscr{K}_{K^\alpha}(\bar{w})}} < \frac{1}{\sqrt{-\mathscr{K}_{K^\alpha}(\bar{w})}}.$$

In fact, given any character α in $\mathbb{T}^n$ and any λ, satisfying $0 < |\lambda| < \frac{1}{\sqrt{(-\mathscr{K}_{K^\alpha}(\bar{w}))}}$, if we choose $a = \frac{\lambda}{\sqrt{1+|\lambda|^2\mathscr{K}_{K^\alpha}(\bar{w})}}$, then clearly we have $\frac{a}{\sqrt{1-|a|^2\mathscr{K}_{K^\alpha}(\bar{w})}} = \lambda$. Hence, given any λ, satisfying $0 < |\lambda| < \frac{1}{\sqrt{J_\Omega(w)}}$ and any character α, we can find a scalar a for which the module $\mathbb{C}^2_w(\lambda)$ is isomorphic to the module $\mathbb{C}^2_w(\frac{a}{\sqrt{1-|a|^2\mathscr{K}_{K^\alpha}(\bar{w})}})$. Equivalently, the module $\mathbb{C}^2_w(\lambda)$ is isomorphic to the module tensor product $\mathscr{H}_{K^\alpha} \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_w(a)$.

Distinct eigenvalue case: Now consider the Hilbert module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ over the algebra $\mathscr{O}(\bar{\Omega})$ and assume that it is a contractive module. Using lemma 5.7, we have $|\lambda| \le \frac{\sqrt{1-(m_\Omega(w_1,w_2))^2}}{m_\Omega(w_1,w_2)} = t_\Omega(w_1,w_2)$. Again, we have two distinct possibilities.

Case 1: $|\lambda| \in \mathscr{I}_\Omega(w_1,w_2)$, see (5.13) for the definition of $\mathscr{I}_\Omega(w_1,w_2)$. In this case, we have the existence of a character α such that $|\lambda| = |\lambda_{w_1,w_2}(\alpha)|$. So, the module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda_{w_1,w_2}(\alpha))$. Equivalently, the Hilbert module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ is isomorphic to the quotient module $\mathscr{S}^\alpha_{w_1,w_2}$ of the Hilbert module $\mathscr{H}_{K^\alpha}$ defined in (5.11).

Case 2: When $|\lambda| \le t_\Omega(w_1,w_2)$ but $|\lambda| \notin \mathscr{I}_\Omega(w_1,w_2)$. From (5.4), we find that the module tensor product $\mathscr{H}_{K^\alpha} \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_{w_1,w_2}(a(w_2-w_1))$ is equal to $\mathbb{C}^2_{w_1,w_2}(\hat{a}(w_2-w_1))$, where $\hat{a}$ is given by

$$\hat{a} = \frac{aK^\alpha_{2,1}}{\sqrt{K^\alpha_{1,1}K^\alpha_{2,2} + |a|^2(K^\alpha_{1,1}K^\alpha_{2,2} - |K^\alpha_{1,2}|^2)}}.$$

Notice that $|\hat{a}| < |\lambda_{w_1,w_2}(\alpha)|$. In fact, given any character α in $\mathbb{T}^n$ and any λ, satisfying $0 < |\lambda| < \lambda_{w_1,w_2}(\alpha)|$, if we choose

$$a = \frac{\lambda\sqrt{K^\alpha_{1,1}K^\alpha_{2,2}}}{\sqrt{|K^\alpha_{2,1}|^2 - |\lambda|^2(K^\alpha_{1,1}K^\alpha_{2,2} - |K^\alpha_{1,2}|^2)}},$$

then clearly we have $|\hat{a}| = |\lambda|$. Hence, given any λ, satisfying $0 < |\lambda| < p_\Omega(w_1,w_2)$ and any character α, we can find a scalar a for which the module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\lambda)$ is isomorphic to the module $\mathbb{C}^2_{w_1,w_2}((w_2-w_1)\hat{a})$. Equivalently, the module $\mathbb{C}^2_{w_1,w_2}(\lambda)$ is isomorphic to the module tensor product $\mathscr{H}_{K^\alpha} \otimes_{\mathscr{O}(\bar{\Omega})} \mathbb{C}^2_{w_1,w_2}(a)$.

5.4.1 Minimality and Non-uniqueness

Let $\mathscr{M}$ be a Hilbert module over $\mathscr{O}(\overline{\Omega})$. Assume that $\mathscr{M}$ admits a boundary dilation. So, there exist a contractive Hilbert module $\mathscr{H}$ over $\mathscr{C}(\partial\Omega)$ such that $P_{\mathscr{M}}\rho_{\mathscr{H}}(f)_{|\mathscr{M}} = \rho_{\mathscr{M}}(f)$ for all $f \in \mathscr{O}(\overline{\Omega})$, where $\rho_{\mathscr{M}}$ and $\rho_{\mathscr{K}}$ are the homomorphism associated with the module $\mathscr{M}$ and $\mathscr{H}$ respectively. Then by Sarason's theorem (see Theorem 5.11), there are submodules $\mathscr{H}_0$ and $\mathscr{H}_1$ of the module $\mathscr{H}$ over the algebra $\mathscr{O}(\overline{\Omega})$ such that $\mathscr{M}$ is the quotient of $\mathscr{H}_0$ by $\mathscr{H}_1$. Thus we have the following exact sequence

$$0 \longrightarrow \mathscr{H}_1 \xrightarrow{\theta} \mathscr{H}_0 \xrightarrow{\psi} \mathscr{M} \longrightarrow 0,$$

where θ and ψ are partial isometric module maps. This is also called as Ŝilov resolution for the module $\mathscr{M}$. Without loss of generality, we may assume that $\mathscr{H}$ is the unique minimal extension of $\mathscr{H}_0$ over the algebra $\mathscr{C}(\partial\Omega)$. Then the dilation of $\mathscr{M}$, or equivalently, the Ŝilov resolution for $\mathscr{M}$ is said to be minimal if the closure of $\{\rho_{\mathscr{K}}(f)(\mathscr{M}) : f \in \mathscr{C}(\partial\Omega)\}$ is equal to $\mathscr{H}$, that is, $\mathscr{H}$ does not have any proper submodule over $\mathscr{C}(\partial\Omega)$ containing $\mathscr{M}$.

The module $\mathscr{S}_w^\alpha$ described in (5.6) admits the resolution

$$0 \longrightarrow \mathscr{H}_1 \xrightarrow{\theta} \mathscr{H}_0 \xrightarrow{\psi} \mathscr{S}_w^\alpha \longrightarrow 0,$$

where $\mathscr{H}_1 = \mathscr{S}_w^{\alpha\perp}$, $\mathscr{H}_0 = H^2_\alpha$ and the maps θ, ψ are the inclusion and the projection maps respectively.

We note that $\int_{\partial\Omega} \log|f| ds < \infty$, for all $f \in H^2_\alpha$, (see [20, Ch.4,Proposition 6.7]). Therefore L^2_α has no proper reducing subspace containing a non trivial subspace of H^2_α since every proper reducing subspace of L^2_α must vanish on a set of positive measure. It now follows that the resolution of $\mathscr{S}_w^\alpha$ is minimal.

Let $\mathscr{M}$ be a contractive module over $\mathscr{O}(\overline{\Omega})$. Suppose $0 \longrightarrow \mathscr{B}_i \longrightarrow \mathscr{P}_i \longrightarrow \mathscr{M} \longrightarrow 0$ are two Ŝilov resolution for the module $\mathscr{M}, i = 1, 2$. We say that these resolutions are isomorphic if there exist module isomorphism $\varphi : \mathscr{B}_1 \mapsto \mathscr{B}_2$ and $\vartheta : \mathscr{P}_1 \mapsto \mathscr{P}_2$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{B}_1 & \longrightarrow & \mathscr{P}_1 & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow \\
0 & \longrightarrow & \mathscr{B}_2 & \longrightarrow & \mathscr{P}_2 & \longrightarrow & \mathscr{M} & \longrightarrow & 0
\end{array}$$

commutes.

In the case of $\Omega = \mathbb{D}$, It is well known that any two minimal Ŝilov resolution for a contractive module $\mathscr{M}$ over $\mathscr{O}(\overline{\mathbb{D}})$ are isomorphic (cf. [19, Theorem 3.15]). However this fails in the case of a finitely connected domain with connectivity greater than 1. (cf. [19, Example 3.9]). We provide another class of example which illustrate this fact.

For a fixed w in Ω, we have proved that the function $\phi : \mathbb{T}^n \to \mathbb{R}$ defined by $\phi(\alpha) = \mathscr{K}_{K^\alpha}(\bar{w})$ is continuous (See Lemma 5.22). It follows that ϕ is never injective. (Indeed, if we assume to the contrary that such a continuous map exists, then one must conclude that the n torus $\mathbb{T}^n$ is homeomorphic to a compact interval leading to a contradiction.) Hence there exists two distinct character α and β so that

$$\mathscr{K}_{K^\alpha}(\bar{w}) = \mathscr{K}_{K^\beta}(\bar{w}).$$

Consequently the quotient modules $\mathscr{S}_w^\alpha$ and $\mathscr{S}_w^\beta$ are isomorphic but the Ŝilov resolution for the modules $\mathscr{S}_w^\alpha$ and $\mathscr{S}_w^\beta$ are not isomorphic.

In the language of Ball, (cf. [6]) this gives an example of two weakly equivalent but strongly inequivalent model for C_{00} completely contractive unital representation over the function algebra $\mathrm{Rat}(\overline{\Omega})$.

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